Riemann Sums

Recall that we have previously discussed the area problem. In its simplest form we can state it this way:

The Area Problem. Let f be a continuous, non-negative function on the closed interval $[a, b]$. Find the area bounded above by $f(x)$, below by the x-axis, and by the vertical lines $x = a$ and $x = b$. See Figure 1.1. To solve this problem we will need to use

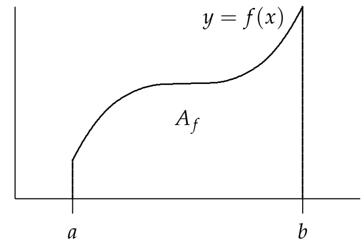

Figure 1.1: Find the area A_f under a nonnegative continuous curve on the interval $[a, b]$.

Basic Area Properties (Axioms). We assume the following properties.

1. The area of a region A is a non-negative real number: $\text{Area}(A) \geq 0$.
2. If A is a subset of B, then $\text{Area}(A) \leq \text{Area}(B)$.
3. If A is subdivided into two non-overlapping regions A_1 and A_2, then
$$\text{Area}(A) = \text{Area}(A_1) + \text{Area}(A_2).$$
4. The area of a rectangle is $b \times h$.

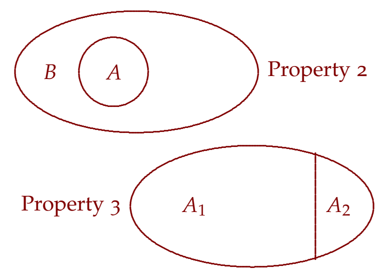

Figure 1.2: Properties 2 and 3.

YOU TRY IT 1.1. Using the area properties above, prove that the area of any triangle is
$$\frac{1}{2}(b \times h).$$
See Figure 1.3. Which area properties do you use in your proof?

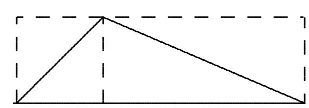

Figure 1.3: Show $A = \frac{1}{2}(b \times h)$.

YOU TRY IT 1.2. How could you use the area formula for a triangle to find the area of any polygon? (See Figure 1.4.) What area properties are used to do this?

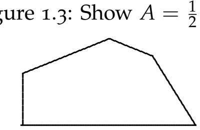

Figure 1.4: How you can find the area of this polygon?

What about curved figures like (semi)circles. Why is the area of a circle πr^2 or, equivalently, the area of a semi-circle $\frac{1}{2}\pi r^2$? If we can solve the general area problem, then we will be able to *prove* that the area of a semi-circle is $\frac{1}{2}\pi r^2$ because we know that the graph of the semi-circle of radius r is given by the continuous, non-negative function $f(x) = \sqrt{r^2 - x^2}$. In other words, a semi-circular region satisfies the conditions outlined in the general area problem.

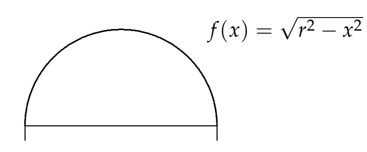

Figure 1.5: This semi-circle satisfies the conditions of the area problem.

Note: We'll solve the area problem two ways. Since the answer must be the same, this equality will be the proof for the so-called Fundamental Theorem of Calculus.

To solve the area problem, we'll need to use the only area formula we know... we must use rectangle regions.

Riemann Sums (Theory)

The presentation here is slightly different than in your text. Make sure that you understand what all of the notation means. Again, remember what we are trying to solve:

The Area Problem. Let f be a continuous, non-negative function on the closed interval $[a, b]$. Find the area bounded above by $f(x)$, below by the x-axis, and by the vertical lines $x = a$ and $x = b$.

As we have just noted, since the only area formula we have have to work with is for rectangles, *we must use rectangles* to approximate the area under the curve. Here's how we go about this approximation process.

Step 1. First subdivide or **partition** $[a, b]$ by choosing points $\{x_0, x_1, \ldots, x_n\}$ where

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

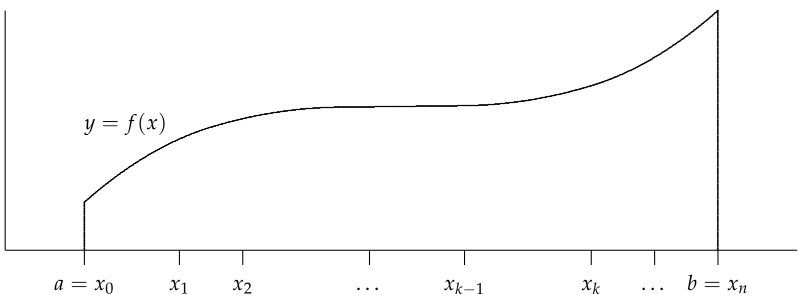

Figure 1.6: A partition of the interval $[a, b]$.

Step 2. Determine the **height** of the kth rectangle by choosing a sample point c_k in the kth subinterval so that $x_{k-1} \le c_k \le x_k$. Use $f(c_k)$ as the height.

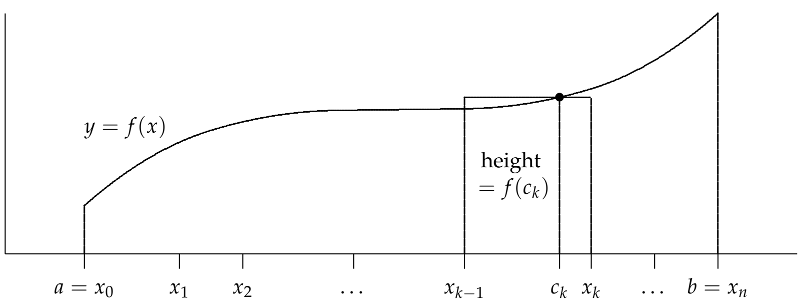

Figure 1.7: $f(c_k)$ is the height of the kth rectangle (see the point marked with a • on the curve).

Step 3. The **width** of the base of the kth rectangle is just $x_k - x_{k-1}$. We usually call this number Δx_k. (See Figure 1.8.)

Step 4. So using the rectangle area assumption, the **area** of the kth rectangle is $h \times b = f(c_k) \times \Delta x_k$.

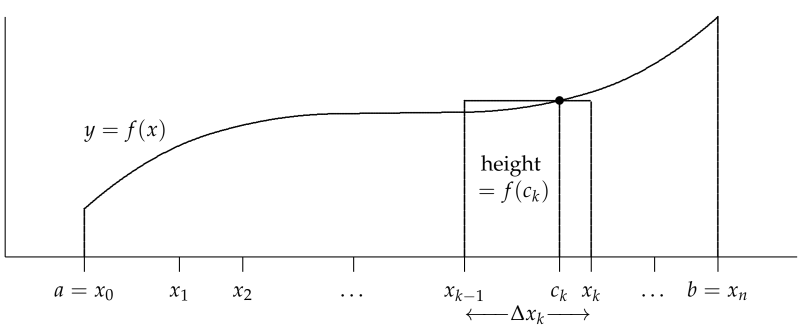

Figure 1.8: $\Delta x_k = x_k - x_{k-1}$ is the width of the kth rectangle. So the area of the kth rectangle is $f(c_k) \times \Delta x_k$.

Step 5. If we carry out this same process for each subinterval determined by the partition $\{x_0, x_1, \ldots, x_n\}$, we get n rectangles. The area under f on $[a, b]$ is *approximately* the sum of the areas of all n rectangles,

$$\text{Area}(A) \approx \sum_{k=1}^{n} f(c_k)\Delta x_k.$$

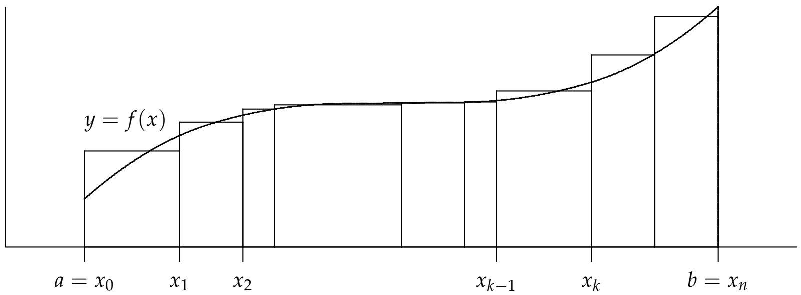

Figure 1.9: A rectangular approximation to the area under f on the interval $[a, b]$.

DEFINITION 1.1 (Riemann Sum). Suppose f is defined on the interval $[a, b]$ with partition $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$. Let $\Delta x_k = x_k - x_{k-1}$ and let c_k be any point chosen so that $x_{k-1} \le c_k \le x_k$. Then

$$\sum_{k=1}^{n} f(c_k)\Delta x_k$$

is called a **Riemann sum** for f on $[a, b]$.

Notice that in the general definition of a Riemann sum we have not assumed that f is non-negative or that it is continuous. The definition makes sense as long as f is defined at every point in $[a, b]$. Let's work out a simple example.

EXAMPLE 1.0.1. Estimate the area under $f(x) = (x-1)^3 + 1$ on the interval $[0, 2]$ using the partition points

$$x_0 = 0$$
$$x_1 = \tfrac{1}{2}$$
$$x_2 = \tfrac{3}{2}$$
$$x_3 = 2$$

and sample points

$$c_1 = \tfrac{1}{2}$$
$$c_2 = 1$$
$$c_3 = \frac{7}{4}$$

SOLUTION. We use Definition 1.1 and form the appropriate Riemann sum. First

$$\Delta x_1 = x_1 - x_0 = \tfrac{1}{2} - 0 = \tfrac{1}{2}$$
$$\Delta x_2 = x_2 - x_1 = \tfrac{3}{2} - \tfrac{1}{2} = 1$$
$$\Delta x_3 = x_3 - x_2 = 2 - \tfrac{3}{2} = \tfrac{1}{2}$$

So

$$\begin{aligned}\text{Area}(A) &\approx \sum_{k=1}^{3} f(c_k)\Delta x_k \\ &= f(\tfrac{1}{2})\Delta x_1 + f(1)\Delta x_2 + f(\tfrac{7}{4})\Delta x_3 \\ &= (\tfrac{7}{8})(\tfrac{1}{2}) + (1)(1) + (\tfrac{91}{64})(\tfrac{1}{2}) \\ &= \frac{275}{128}\end{aligned}$$

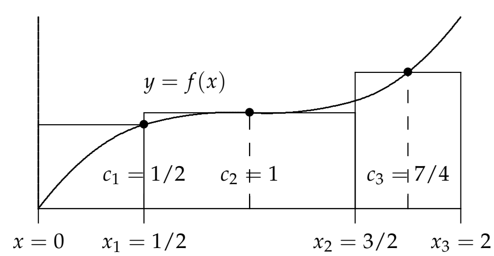

Figure 1.10: A Riemann sum for $f(x) = (x-1)^3 + 1$ on the interval $[0, 2]$ using three rectangles. The height for each rectangle is marked with a $\bullet$. Does the approximation seem to be an under- or overestimate of the true area?

The Riemann sum provides an estimate of $\frac{275}{128}$ as the area under the curve. Yet we don't know how accurate that estimate is and we still don't know the true area under the curve.

Further, notice that the use of summation notation was not particularly helpful here. If we use Riemann sums in a more systematic way, Riemann sum notation can be very helpful. And, if we are careful about how we form such sums, we can even say whether the sum is an over- or underestimate of the actual area under the curve.

Regular Partitions, Upper and Lower Sums

Again let us assume that $y = f(x)$ is a non-negative, continuous function on the interval $[a, b]$. We will now take a more systematic approach to forming Riemann sums for f on $[a, b]$ that will allow us to make more accurate approximations to the area under the curve. Again we proceed in a series of steps.

Step 1. Divide the interval $[a, b]$ into n equal-width subintervals. The width of each interval will be

$$\Delta x = \frac{b-a}{n}.$$

We can express the partition points in terms of a and Δx.

$$\begin{aligned} x_0 &= a = a + 0 \cdot \Delta x \\ x_1 &= a + \Delta x \\ x_2 &= a + 2\Delta x \\ &\vdots \\ x_k &= a + k\Delta x \\ &\vdots \\ x_n &= b = a + n\Delta x \end{aligned}$$

Equal width partitions are called **regular partitions**. The formula for the kth point in a regular partition is

$$\boxed{x_k = a + k\Delta x} \tag{1.1}$$

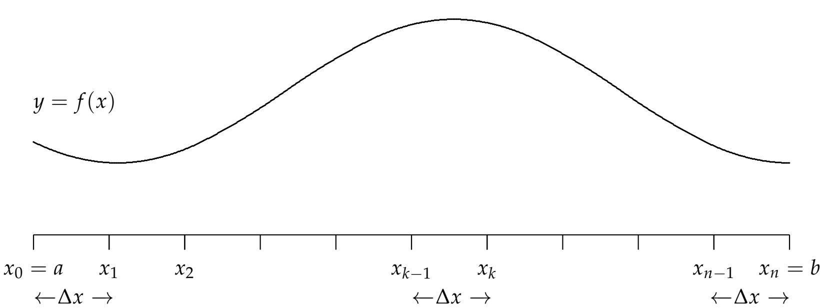

Figure 1.11: A regular partition of the interval $[a, b]$ into n subintervals each of length $\Delta x = \frac{b-a}{n}$. This means that $x_k = a + k\Delta x$.

Step 2. Since f is continuous, it achieves a maximum value and a minimum value on each subinterval. We use the following notation to represent these points.

$$\boxed{f(M_k) = \text{maximum value of } f \text{ on the } k\text{th subinterval}}$$

$$\boxed{f(m_k) = \text{minimum value of } f \text{ on the } k\text{th subinterval}}$$

These points are illustrated in Figure 1.12

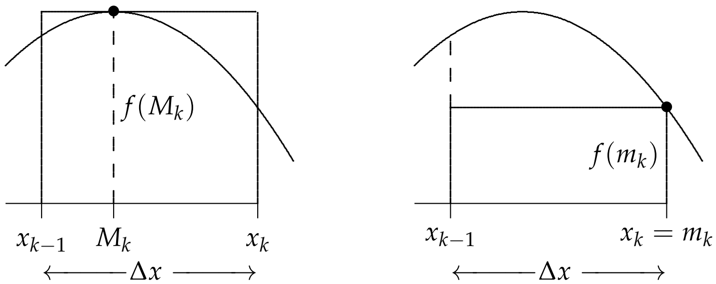

Figure 1.12: On the kth subinterval the maximum height $f(M_k)$ occurs between the two endpoints. The minimum height $f(m_k)$ happens to occur at the right endpoint of the interval, $m_k = x_k$.

Figure 1.12 shows that we get two different rectangles for each subinterval depending on whether we choose the maximum or the minimum value of f as the height. These are called the circumscribed and inscribed rectangles, respectively. We see that

$$\boxed{\text{area of the } \textit{circumscribed} \text{ rectangle} = f(M_k)\Delta x}$$

$$\boxed{\text{area of the } \textit{inscribed} \text{ rectangle} = f(m_k)\Delta x}$$

Step 3. To obtain an approximation for the area under the curve, we form a Riemann sum using either the circumscribed (upper) or inscribed (lower) rectangles.

If we add up all the circumscribed rectangles for a regular partition with n subintervals we get the **upper sum** for the partition:

$$\boxed{\text{Upper Riemann Sum} = \text{Upper}(n) = \sum_{k=1}^{n} f(M_k)\Delta x.} \tag{1.2}$$

If we add up all the inscribed rectangles for a regular partition we get the **lower sum** for the partition:

$$\boxed{\text{Lower Riemann Sum} = \text{Lower}(n) = \sum_{k=1}^{n} f(m_k)\Delta x.} \tag{1.3}$$

Take a moment to review all of the notation. Ok? Let's see how these upper and lower sums are computed in a simple case.

EXAMPLE 1.0.2. Let $y = f(x) = 1 + \frac{1}{2}x^2$ on $[0,2]$. Determine Upper(4) and Lower(4), the upper and lower Riemann sums for a regular partition into four subintervals.

SOLUTION. We use the steps outlined above.

Step 1. Determine Δx. Here $[a,b] = [0,2]$ and $n = 4$ so

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}.$$

Step 2. Determine the partition points, x_k. Using (1.1)

$$x_k = a + k\Delta x = 0 + k\left(\frac{1}{2}\right) = \frac{k}{2}. \tag{1.4}$$

Step 3. Take a look at the graph of $f(x) = 1 + \frac{1}{2}x^2$ on $[0,2]$ in Figure 1.13. Since f is an increasing function, the maximum value of f on each subinterval occurs at the right-hand endpoint of the interval. The right-hand endpoint of the i interval is just x_k. So

$$M_k = x_k = \frac{k}{2}.$$

Consequently, the maximum value of f on the kth interval is

$$f(M_k) = f\left(\frac{k}{2}\right) = 1 + \frac{1}{2}\left(\frac{k}{2}\right)^2 = 1 + \frac{k^2}{8}.$$

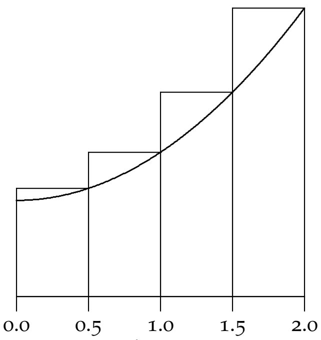

Figure 1.13: The upper sum Upper(4) for the function $f(x) = 1 + \frac{1}{2}x^2$ on $[0,2]$. The maximum value of the function occurs at the right-hand endpoint, x_k for each subinterval.

Step 4. Putting this all together, the upper Riemann sum is

$$\text{Upper}(4) = \sum_{k=1}^{4} f(M_k)\Delta x = \sum_{k=1}^{4} f\left(\frac{k}{2}\right)\frac{1}{2} = \sum_{k=1}^{4}\left[1 + \frac{k^2}{8}\right]\frac{1}{2}$$

Now use the basic summation rules and formulæ to evaluate the sum.

$$\begin{aligned}\text{Upper}(4) = \sum_{k=1}^{4}\left[1 + \frac{k^2}{8}\right]\frac{1}{2} &= \frac{1}{2}\sum_{k=1}^{4} 1 + \frac{1}{16}\sum_{k=1}^{4} k^2 \\ &= \frac{1}{2}[4(1)] + \frac{1}{16}\left(\frac{4(5)(9)}{6}\right) \\ &= \frac{31}{8}.\end{aligned}$$

The lower sum Lower(4) can be calculated in a similar way. Again, because the function is increasing, the minimum value of f on the kth subinterval occurs at the left-hand endpoint x_{k-1}. Using the formula in (1.4)

$$m_k = x_{k-1} = \frac{k-1}{2}.$$

Consequently, the minimum value of f on the kth interval is

$$f(m_k) = f\left(\frac{k-1}{2}\right) = 1 + \frac{1}{2}\left(\frac{k-1}{2}\right)^2 = 1 + \frac{k^2 - 2i + 1}{8} = \frac{9}{8} - \frac{k}{4} + \frac{k^2}{8}.$$

Putting this all together, the lower Riemann sum is

$$\text{Lower}(4) = \sum_{k=1}^{4} f(m_k)\Delta x = \sum_{k=1}^{4}\left[\frac{9}{8} - \frac{k}{4} + \frac{k^2}{8}\right]\frac{1}{2}$$

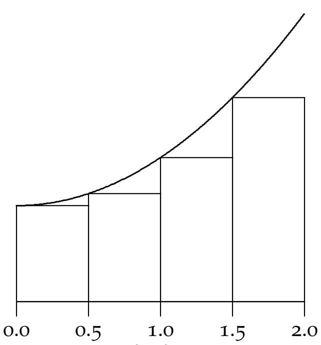

Figure 1.14: The lower sum Lower(4) for the function $f(x) = 1 + \frac{1}{2}x^2$ on $[0,2]$. The minimum value of the function occurs at the left-hand endpoint, x_{k-1} for each subinterval.

Again use the basic summation rules and formulæ to evaluate the sum.

$$\begin{aligned}\text{Lower}(4) = \sum_{k=1}^{4}\left[\frac{9}{8} - \frac{k}{4} + \frac{k^2}{8}\right]\frac{1}{2} &= \frac{1}{2}\sum_{k=1}^{4}\frac{9}{8} - \frac{1}{8}\sum_{k=1}^{4} k + \frac{1}{16}\sum_{k=1}^{4} k^2 \\ &= \frac{1}{2}\left[4\left(\frac{9}{8}\right)\right] - \frac{1}{8}\left(\frac{4(5)}{2}\right) + \frac{1}{16}\left(\frac{4(5)(9)}{6}\right) \\ &= \frac{23}{8}.\end{aligned}$$

The advantage of upper and lower sums is that the true area under the curve is trapped between their values. Upper(n) is always an overestimate and Lower(n) is an underestimate. More precisely,

$$\boxed{\text{Lower}(n) \le \text{area under } f \le \text{Upper}(n).}$$

In this example,

$$\text{Lower}(4) = \frac{23}{8} \le \text{area under } f \le \frac{31}{8} = \text{Upper}(n).$$

Here are two questions to think about: How can we improve the estimate? Which sum was easier to compute, the lower or the upper? Why?

Now let's do the whole process again. This time, though we will use n subintervals, without specifying what the actual value of n is. This is where the summation notation that we have developed really comes to the rescue.

EXAMPLE 1.0.3. Let $y = f(x) = 1 + \frac{1}{2}x^2$ on $[0, 2]$. Determine Upper(n) and Lower(n), the upper and lower Riemann sums for a regular partition into n subintervals.

SOLUTION. Step 1. Determine Δx. Here $[a, b] = [0, 2]$ so

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}.$$

Step 2. Determine the partition points, x_k. Using (1.1)

$$x_k = a + k\Delta x = 0 + k\left(\frac{2}{n}\right) = \frac{2k}{n}. \tag{1.5}$$

Step 3. Since f is an increasing function, the maximum value of f on each subinterval occurs at the right-hand endpoint of the interval. So $M_k = x_k$. So

$$M_k = x_k = \frac{2k}{n}.$$

Consequently, the maximum value of f on the kth interval is

$$f(M_k) = f\left(\frac{2k}{n}\right) = 1 + \frac{1}{2}\left(\frac{2k}{n}\right)^2 = 1 + \frac{4k^2}{2n^2} = 1 + \frac{2k^2}{n^2}.$$

Putting this all together, the upper Riemann sum is

$$\begin{aligned}
\text{Upper}(n) = \sum_{k=1}^{n} f(M_k)\Delta x = \sum_{k=1}^{n}\left[1 + \frac{2k^2}{n^2}\right]\frac{2}{n} &= \frac{2}{n}\sum_{k=1}^{n} 1 + \frac{4}{n^3}\sum_{k=1}^{n} k^2 \\
&= \frac{2}{n}[n(1)] + \frac{4}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \\
&= 2 + \frac{2}{3}\left(\frac{2n^2+3n+1}{n^2}\right) \\
&= 2 + \left(\frac{4}{3} + \frac{2}{n} + \frac{2}{3n^2}\right) \\
&= \frac{10}{3} + \frac{2}{n} + \frac{2}{3n^2}
\end{aligned}$$

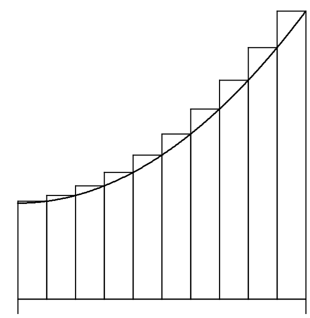

Figure 1.15: The upper sum Upper(n) for the function $f(x) = 1 + \frac{1}{2}x^2$ on $[0, 2]$. As n increases, Upper(n) better approximates the area under the curve. (Compare to Figure 1.12.)

The lower sum Lower(n) can be calculated in a similar way. The minimum value of f on the kth subinterval occurs at the left-hand endpoint:

$$m_k = x_{k-1} = \frac{2(k-1)}{n}.$$

The minimum value of f on the kth interval is

$$f(m_k) = f\left(\frac{2(k-1)}{n}\right) = 1 + \frac{1}{2}\left(\frac{2(k-1)}{n}\right)^2 = 1 + \frac{4(k^2-2k+1)}{2n^2} = 1 + \frac{2(k^2-2k+1)}{n^2}.$$

Putting this all together, the lower Riemann sum is

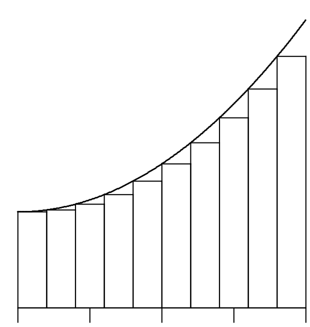

Figure 1.16: The lower sum Lower(n) for the function $f(x) = 1 + \frac{1}{2}x^2$ on $[0, 2]$. The lower sum is an underestimate of the area under f.

$$\begin{aligned}
\text{Lower}(n) = \sum_{k=1}^{n} f(m_k)\Delta x &= \sum_{k=1}^{n}\left[1+\frac{2(k^2-2k+1)}{n^2}\right]\frac{2}{n} \\
&= \frac{2}{n}\sum_{k=1}^{n}1 + \frac{4}{n^3}\sum_{k=1}^{n}(k^2-2k+1) \\
&= \frac{2}{n}[n(1)] + \frac{4}{n^3}\sum_{k=1}^{n}k^2 - \frac{8}{n^3}\sum_{k=1}^{n}k + \frac{4}{n^3}\sum_{k=1}^{n}1 \\
&= 2 + \frac{4}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] - \frac{8}{n^3}\left[\frac{n(n+1)}{2}\right] + \frac{4}{n^3}\left[(n)1\right] \\
&= 2 + \left[\frac{4}{3}+\frac{2}{n}+\frac{2}{3n^2}\right] - \left[\frac{4}{n}+\frac{4}{n^2}\right] + \frac{4}{n^2} \\
&= \frac{10}{3} - \frac{2}{n} + \frac{2}{3n^2}.
\end{aligned}$$

We know that

$$\text{Lower}(n) \le \text{area under } f \le \text{Upper}(n).$$

The formulæ for Upper(n) and Lower(n) are valid for all positive integers n. We expect that as n increases the approximations improve. In this case, taking limits

$$\lim_{n\to\infty}\text{Lower}(n) \le \text{area under } f \le \lim_{n\to\infty}\text{Upper}(n),$$

equivalently,

$$\lim_{n\to\infty}\left[\frac{10}{3}-\frac{2}{n}+\frac{2}{3n^2}\right] \le \text{area under } f \le \lim_{n\to\infty}\left[\frac{10}{3}+\frac{2}{n}+\frac{2}{3n^2}\right],$$

or

$$\frac{10}{3} \le \text{area under } f \le \frac{10}{3}.$$

The only way this can happen is if

$$\text{area under } f = \frac{10}{3}.$$

Take-home Message. This is great! We have managed to determine the area under an actual curve by using approximations by lower and upper Riemann sums. The approximations improve as n increases. By taking limits we hone in on the precise area. This is more carefully described in Theorem 1.1 at the beginning of the next section.

Finally, again ask yourself which of the two sums was easier to calculate? Why was it easier? Shortly we will take advantage of this situation.

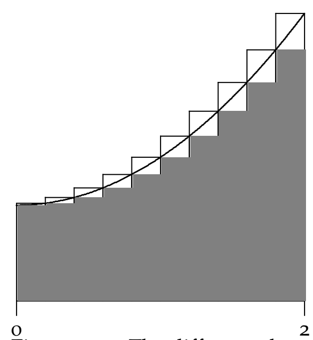

Figure 1.17: The difference between the upper sum Upper(n) for the function $f(x) = 1 + \frac{1}{2}x^2$ on $[0, 2]$ and the lower sum Lower(n) (shaded). The true area lies between the two.