This homework, on Sections 3.1 and 3.2, is due on Monday, October 19.
(There will be a test in class on Wednesday, October 21.)

1. Show that every subset of a countably infinite set is either finite or countably infinite.

2. Use a proof by contradition to prove the following: Suppose X is a countably infinite set and A is a finite subset of X. Then $X \smallsetminus A$ is countably infinite. (You will need the result from the previous problem.)

3. We know that subsets of the set $\{0, 1, ..., n-1\}$ correspont to sets of n-bit binary numbers. This is a one-to-one correspondence between the power set of $\{0, 1, ..., n-1\}$ and the set of n-bit binary numbers.

Consider the power set of the set of natural numbers $\mathbb{N} = \{0, 1, 2, 3, \dots\}$, and let B be the set of all non-negative (binary) integers. (Yes, B and $\mathbb{N}$ are really the same set, but it's useful to have two different names.) We can define a function $f\colon B \to \mathcal{P}(\mathbb{N})$ in the same way that we do in the finite case. That is, for $b \in B$, we define $f(b)$ to be the subset of $\mathbb{N}$ which contains exactly those numbers whose corresponding bit position in B contains a 1. For example, $f(10110001001_2) = \{0, 3, 7, 8, 10\}$.

- **a)** f is not a one-to-one correspondence. Why not? Exactly which subsets of $\mathbb{N}$ are of the form $f(b)$ for some $b \in B$, and which are not of that form? Why? (Hint: Read part **b**.)
- **b)** What does this say about the size of the set of *finite* subsets of $\mathbb{N}$? What does it say about the size of the set of *infinite* subsets of $\mathbb{N}$? Why?

4. Suppose that L and M are languages over the alphabet $\Sigma = \{a, b\}$, defined as follows:

$$L = \{a, aa\} \qquad M = \{w \in \Sigma^* \mid w \text{ ends with } b\}$$

Identify the following languages. (Write the language out as a set, or give a clear English description of the language; explain your reasoning if the answer is not obvious.)

a) $L \cap M$ **b)** $L \cup M$ **c)** L^3 **d)** L^* **e)** ML
f) LM **g)** M^* **h)** $\overline{M}$ **i)** M^R **j)** $M^R M$

5. Give an English description of the language generated by each of the following regular expressions over the alphabet $\{a, b\}$, or write out the answer as a set:

a) bab^* **b)** $b(ab)^*$ **c)** $(a|b)^*bbb(a|b)^*$
d) $a^*ba^*ba^*ba^*$ **e)** $a^*(b|\varepsilon)a^*(b|\varepsilon)a^*(b|\varepsilon)a^*$

6. Find a regular expression that generates each of the following languages over the alphabet $\{0, 1\}$, and explain in words how your regular expression works.

- **a)** $\{w \in \Sigma^* \mid |w| \geq 2 \text{ and } w \text{ starts and ends with the same symbol}\}$
- **b)** $\{w \in \Sigma^* \mid \text{ every 1 in } w \text{ is immediately followed by a 0}\}$
- **c)** $\{w \in \Sigma^* \mid |w| \text{ is an even number}\}$

7. Find a regular expression that generates each of the following languages over the alphabet $\Sigma = \{a, b, c\}$, and explain in words how your regular expression works.

- **a)** $\{w \in \Sigma^* \mid n_a(w) \text{ is an odd number}\}$
- **b)** $\{w \in \Sigma^* \mid w \text{ contains the string } ab\}$
- **c)** $\{w \in \Sigma^* \mid w \text{ does } \textbf{not} \text{ contain the string } ab\}$