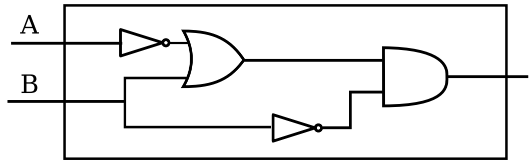
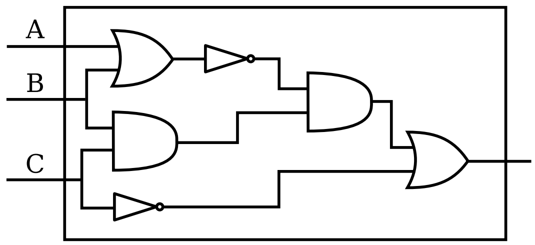
1. (*3 points*) Find the Boolean expression that gives the output of each circuit as a function of its inputs. (Show your work by redrawing the circuit and labeling the output of each logic gate.)

a)

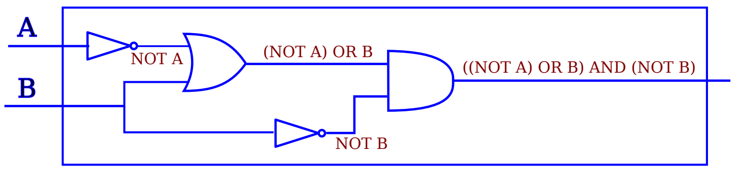


b)

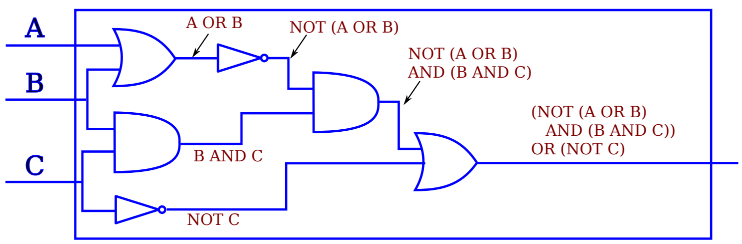


Answer:

a)



b)



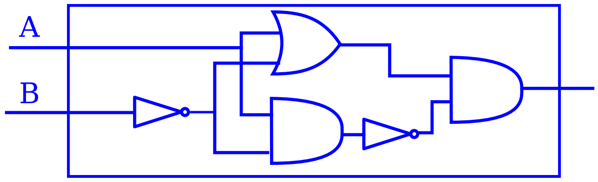
2. (*3 points*) Draw a logic circuit that computes each of the following Boolean expressions:

a) $(A \vee (\neg B)) \wedge (\neg(A \wedge (\neg B)))$

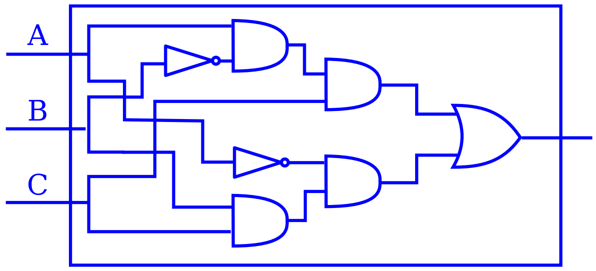
b) $(A \wedge (\neg B) \wedge C) \vee ((\neg A) \wedge B \wedge C)$

Answer:

a)



b)



3. (*4 points*) Express the following statements in predicate logic. Try to express as much of the meaning as you can. Give the meaning of each predicate that you use. If it is not clear what the domain of discourse is for a predicate, state the domain of discourse explicitly.

a) Every elephant is gray.

b) There is a pink elephant.

c) Everyone owns a pink elephant.

d) There is a city where all elephants are pink.

Answer:

I will use the following predicates: $E(x)$ means x is an elephant. $G(x)$ means x is grey. $P(x)$ means x is pink. $C(x)$ means x is a city. $L(x, y)$ means x is located in y, where the Domain of Discourse for y is locations. $O(x, y)$ means x owns y, where the Domain of Discourse for x is people, and the Domain of Discourse for y is things. (Where the Domain of Discourse isn't mentioned, it's all entities.)

a) $\forall x(E(x) \rightarrow G(x))$

b) $\exists x(E(x) \wedge P(x))$

c) $\forall x\, \exists y\big(P(x) \wedge E(x) \wedge O(x, y)\big)$

d) $\exists x\big(\, C(x) \wedge \forall y\big((E(y) \wedge L(y, x)) \rightarrow P(y)\big)\,\big)$

4. (*2 points*) The following two Boolean expressions are not logically equivalent. Explain the difference in the meaning of the two expressions. Give an example of two specific predicates for which one of the expressions is true while the other one is false. (One possibility is two predicates for which the domain of discourse is integers.)

$$\big(\forall x, P(x)\big) \vee \big(\forall x, Q(x)\big) \qquad \text{and} \qquad \forall x\big(P(x) \vee Q(x)\big)$$

Answer:

For the first expression to be true, either $P(x)$ must be true for all x, or $Q(x)$ must be true for all x. For the second to be true, it is possible that $P(x)$ is true for some x while $Q(x)$ is true for other x—if $P(x)$ is false for some x, then $Q(x)$ must be true for that x in order to make $P(x) \vee Q(x)$ true.

As an example, suppose that $P(x)$ is $x < 10$ and $Q(x)$ is $x > 0$, where the domain of discourse is all real numbers. For any real number x, at least one of $P(x)$ or $Q(x)$ is true, so $\forall x(P(x) \vee Q(x))$ is true. (Note that for some x, both $P(x)$ and $Q(x)$ are true, but that is not actually relevant here.) However, $\forall x P(x)$ is false because not every number is less than ten, and $\forall x Q(x)$ is false because not every number is greater than zero. So for this example, $\big(\forall x, P(x)\big) \vee \big(\forall x, Q(x)\big)$ is false, but $\forall x\big(P(x) \vee Q(x)\big)$ is true.

5. (*4 points*) Simplify each of the following expressions. Simplify the answer, so that the operator $\neg$ is only applied to individual predicates. (Show your work by writing a chain of logical equivalences, starting from the given expression.)

a) $\neg\big(\,\forall x\big(P(x) \vee Q(x)\big)\,\big)$

b) $\neg\big(\,\forall x(P(x) \rightarrow (Q(x) \wedge R(x)))\,\big)$

c) $\neg\big(\,\exists y(H(y) \wedge \forall x L(x,y))\,\big)$

d) $\neg\big(\,\forall x\,\exists y\,\exists z\,(L(x,y) \wedge G(x,z))\,\big)$

Answer:

$$\textbf{(a)}\quad \neg\big(\,\forall x\big(P(x) \vee Q(x)\big)\,\big) \equiv \exists x\big(\neg(P(x) \vee Q(x))\big) \equiv \exists x\big((\neg P(x)) \wedge (\neg Q(x))\big)$$

$$\textbf{(b)}\quad \neg\big(\,\forall x(P(x) \rightarrow (Q(x) \wedge R(x)))\,\big) \equiv \exists x\big(\neg(P(x) \rightarrow (Q(x) \wedge R(x)))\big) \equiv \exists x\big(P(x) \wedge \neg(Q(x) \wedge R(x))\big) \equiv \exists x\big(P(x) \wedge ((\neg Q(x)) \vee (\neg R(x)))\big)$$

$$\textbf{(c)}\quad \neg\big(\,\exists y(H(y) \wedge \forall x L(x,y))\,\big) \equiv \forall y\big(\neg(H(y) \wedge \forall x L(x,y))\big) \equiv \forall y\big((\neg H(y)) \vee (\neg \forall x L(x,y))\big) \equiv \forall y\big((\neg H(y)) \vee \exists x(\neg L(x,y))\big)$$

$$\textbf{(d)}\quad \neg\big(\,\forall x\,\exists y\,\exists z\,(L(x,y) \wedge G(x,z))\,\big) \equiv \exists x\,\forall y\,\forall z\big(\neg(L(x,y) \wedge G(x,z)\big) \equiv \exists x\,\forall y\,\forall z\big((\neg L(x,y)) \vee (\neg G(x,z))\big)$$

6. (*3 points*) Use a *truth table* to show that the following argument is valid. And then, explain in English why it *makes sense* that this argument is valid. (What does the argument *mean*?)

$$\begin{array}{l} p \rightarrow q \\ (\neg p) \rightarrow q \\ \hline \therefore\ q \end{array}$$

Answer:

We need a truth table to show that $\big((p \rightarrow q) \wedge ((\neg p) \rightarrow q)\big) \rightarrow q$ is a tautology:

p	q	$\neg p$	$p \to q$	$(\neg p) \to q$	$((p \to q) \wedge ((\neg p) \to q)$	$((p \to q) \wedge ((\neg p) \to q)) \to q$
T	T	F	T	T	T	T
T	F	F	F	T	F	T
F	T	T	T	T	T	T
F	F	T	T	F	F	T

Since all of the entries in the last column are true, the expression is a tautology, which means by definition that the argument is valid.

It makes sense that the argument is valid because one of the two cases, p and $\neg p$, has to be true. In a case where p is true, the premise $p \to q$ allows us to deduce q. On the other hand, in a case where $\neg p$ is true, the premise $(\neg p) \to q$ allows us to deduce q. That is, q can be deduced from the premises in all cases.

7. (*6 points*) Give a formal proof for each of the following valid arguments. For each step in the proof, give the justification for that step.

a)
$$\begin{array}{l} p \to q \\ q \to (r \vee s) \\ \neg s \\ p \\ \hline \therefore\ r \end{array}$$

b)
$$\begin{array}{l} (p \wedge q) \to (r \vee s) \\ \neg r \\ p \to q \\ p \\ \hline \therefore\ s \end{array}$$

c)
$$\begin{array}{l} p \to r \\ (r \wedge s) \to t \\ q \to \neg t \\ s \\ q \\ \hline \therefore\ \neg p \end{array}$$

Answer:

(a)
1. $p \to q$ (premise)
2. p (premise)
3. q (from 1 and 2 by Modus Ponens)
4. $q \to (r \vee s)$ (premise)
5. $r \vee s$ (from 3 and 4 by Modus Ponens)
6. $\neg s$ (premise)
7. r (from 5 and 6 by Elimination)

(b)
1. $p \to q$ (premise)
2. p (premise)
3. q (from 1 and 2 by Modus Ponens)
4. $p \wedge q$ (from 2 and 3)
5. $(p \wedge q) \to (r \vee s)$ (premise)
6. $r \vee s$ (from 4 and 5 by Modus Ponens)
7. $\neg r$ (premise)
8. s (from 6 and 7 by Elimination)

(c)

1.	$q \to \neg t$	(premise)
2.	q	(premise)
3.	$\neg t$	(from 1 and 2 by Modus Ponens)
4.	$(r \wedge s) \to t$	(premise)
5.	$\neg(r \wedge s)$	(from 3 and 4 by Modus Tollens)
6.	$(\neg r) \vee (\neg s)$	(from 5 by DeMorgan's Law)
7.	s	(premise)
8.	$\neg(\neg s)$	(from 7 by Double Negation)
9.	$\neg r$	(from 6 and 8 by Elimination)
10.	$p \to r$	(premise)
11.	$\neg p$	(from 9 and 10 by Modus Tollens)