CPSC 229, Fall 2008 — Homework #5

This homework is due in class on Wednesday, October 22. Please note that there is a ***test*** *coming up on Friday, October 24.*

1. Suppose that $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$. Let $f\colon A \to B$ be the function defined by the "graph" or set of ordered pairs $f = \{(a, 2), (b, 1), (c, 2), (d, 3)\}$. Let $g\colon B \to A$ be the function defined by the set of ordered pairs $g = \{(1, d), (2, c), (3, a)\}$. Find the sets of ordered pairs that define the two composition functions $f \circ g$ and $g \circ f$. Show your work.

2. Show that every subset of a countably infinite set is countable. (Remember that "countable" means either finite or countably infinite.)

3. Use a proof by contradition to prove the following: Suppose X is a countably infinite set and A is a finite subset of X. Then $X \smallsetminus A$ is countably infinite.

4. **a)** What is $|\mathcal{P}(\emptyset)|$? Why? (Don't just quote the formula. Why does the formula work in this case?)

b) Make up two sets A and B, and give at least two representative examples of elements of $\mathcal{P}(A \times B)$ and at least two representative examples of elements of $\mathcal{P}(A) \times \mathcal{P}(B)$.

c) Let A and B be finite sets. In general, which has more elements, $\mathcal{P}(A \times B)$ or $\mathcal{P}(A) \times \mathcal{P}(B)$? Is your answer true for all finite sets A and B? If not, what are the exceptions? Justify your answers.

5. We know that the power set of the set $\{0, 1, ..., n-1\}$ is in one-to-one correspondence with the set of n-bit binary numbers. Consider the power set of the set of natural numbers $\mathbb{N} = \{0, 1, 2, 3, \dots\}$, and let B be the set of all non-negative (binary) integers. (Yes, B and $\mathbb{N}$ are really the same set, but it's useful to have two different names.) We can define a function $f\colon B \to \mathcal{P}(\mathbb{N})$ in the same way that we do in the finite case. That is, if $b \in B$, we define $f(b)$ to be the subset of $\mathbb{N}$ which contains exactly those numbers whose corresponding bit position in B is 1. For example, $f(10110001001_2) = \{0, 3, 7, 8, 10\}$.

a) f is not a one-to-one correspondence. Why not? Exactly which subsets of $\mathbb{N}$ are of the form $f(b)$ for some $b \in B$? Why?

b) What does this say about the size of the set of *finite* subsets of $\mathbb{N}$? What does it say about the size of the set of *infinite* subsets of $\mathbb{N}$? Why?

6. Let Σ be the alphabet $\Sigma = \{a, b, c\}$. Let L and M be languages over Σ defined by $L = \{\varepsilon, b, bb\}$ and $M = \{a, ac, acc, accc, acccc, \dots\}$. Find each of the following languages. In each case, either list the elements of the language or give a clear description of the set of strings that make up the language.

a) L^2	**b)** L^3	**c)** ML	**d)** M^2	**e)** $L \cup M$
f) L^*	**g)** M^2	**h)** M^*	**i)** $\overline{L}$	**j)** $L \cap M$

7. The **reverse** of a language K is defined to be the language $K^R = \{x^R \mid x \in K\}$. Find the reverse of each of the languages L and M from the previous problem.

8. Give an example of a language L such that $L = L^*$. Justify your answer.