This homework is due in class on Wednesday, December 10.
(This is the final homework of the semester.)

1. Consider the following grammar over the alphabet $\Sigma = \{a, b\}$, which generates the language $\{\, a^{2^n} \mid n \in \mathbb{N} \,\}$:

$$
\begin{aligned}
S &\longrightarrow TaE \\
T &\longrightarrow TD \\
T &\longrightarrow F \\
Da &\longrightarrow aaD \\
DE &\longrightarrow E \\
Fa &\longrightarrow aF \\
FE &\longrightarrow \varepsilon
\end{aligned}
$$

a) Find a derivation for the string a, using this grammar. (a is a^{2^n} for $n = 0$)

b) Find a derivation for the string aa, using this grammar. (aa is a^{2^n} for $n = 1$)

c) Find a derivation for the string $aaaa$, using this grammar. ($aaaa$ is a^{2^n} for $n = 2$)

d) Explain in words how this grammar works and how the string a^{2^n} can be generated for any $n \in \mathbb{N}$.

2. Find a grammar for the language $\{\, a^{3^n+1} \mid n \in \mathbb{N} \,\}$. (Hint: Use a simple modification of the grammar from the preceding problem.)

3. Find a grammar for the language $\{\, ww \mid w \in \{a, b, c\}^* \,\}$. Explain your plan for the grammar: What are the stages in the construction of a string in the language, and how do the rules in your grammar carry out the construction? (Hint: Along the way to making *abbcaccabbcacc*, you can make $aAbBbBcCaAcCcC$ (with an extra character or two at the beginning and/or end).) Then show a derivation of the string *cbcb* using your grammar.

4. Find a grammar for the language $\{\, a^n b^{2n} c^{3n} \mid n \in \mathbb{N} \,\}$. Explain how your grammar works.

5. Find a grammar for the language $\{\, a^k b^n \mid k, n \in \mathbb{N}$ and n is a multiple of $k \,\}$. Explain how your grammar works. (Hint: This language is similar to $\{\, a^n b^n c^{nm} \mid n, m \in \mathbb{N} \,\}$ without the b's.)

6. Draw a transition diagram for a Turing Machine that decides the language $\{\, w \in \{a, b\}^* \mid n_a(w)$ is an even number $\}$.

7. Let $\Sigma = \{a, b\}$. Draw a transition diagram for a Turing machine that computes the function $f \colon \Sigma^* \to \Sigma^*$ defined by $f(w) = w^R$.