1. The first 7 numbers are 1, 3, 7, 15, 31, 63, 127. These were obtained as follows:

$$\begin{aligned}
A_1 &= 1 \qquad \text{(Given)}\\
A_2 &= 2 \times A_1 + 1 = 2 \times 1 + 1 = 3\\
A_3 &= 2 \times A_2 + 1 = 2 \times 3 + 1 = 7\\
A_4 &= 2 \times A_3 + 1 = 2 \times 7 + 1 = 15\\
A_5 &= 2 \times A_4 + 1 = 2 \times 15 + 1 = 31\\
A_6 &= 2 \times A_5 + 1 = 2 \times 31 + 1 = 63\\
A_7 &= 2 \times A_6 + 1 = 2 \times 63 + 1 = 127
\end{aligned}$$

(Note: In fact, A_n is equal to $2^n - 1$ for all positive integers n.)

2. a) The first 8 triangular numbers are 1, 3, 6, 10, 15, 21, 28, 36. The first five numbers come from counting the dots in the pictures. We see that to go from one picture to the next, we add a new row of dots at the bottom, with one more dot than in the bottom row in the previous picture. To go from 15 to 21, we add a row of 6 dots; since the bottom row in the fifth picture has 5 dots, we add a new row with 6 dots. For the next picture, we add 7 dots, giving 21+7 or 28 dots in the seventh picture. For the eighth picture, we have 28+8 or 36 dots. (Note that instead of thinking of adding rows, you could think that you are adding columns or a new diagonal. In any case, to get the n^{th} picture from the previous picture, you add n new dots.)

b) The first row always has 1 dot, the second row has 2 dots, the third row has 3 dots, and so on. In the n^{th} picture there are n rows, and the numbers of dots in the rows are 1, 2, 3, …, n. So the total number of dots in the n^{th} picture is $1+2+3+\cdots+n$. This sum gives the value of the n^{th} triangular number. Let's give the n^{th} triangular number the name T_n. Then we have:

$$T_n = 1 + 2 + 3 + \cdots + n$$

c) In the picture, we see two copies of T_5, one made of filled-in circles and an upside-down one make of outlines of circles. In the whole rectangle, there are 5 rows and 6 columns of circles, so the total number of circles is 30. Since this is two copies of T_5 we see that $T_5 = \frac{30}{2} = 15$. Now, the 30 is actually $5 \times (5 + 1)$, where the first 5 is the number of rows and the $5 + 1$ comes from the fact that the bottom row is 5 filled circles plus one outline circle, so in fact the formula is $T_5 = \frac{5\times(5+1)}{2}$. We can make a similar picture for T_n: It will have two copies of T_n forming the two halves of a rectangle. The rectangle will have n rows and $n + 1$ columns, for a total of $n(n + 1)$ circles. Since the number of circles in the rectangle is twice T_n, we get that $T_n = \frac{n(n+1)}{2}$. (Note that combining part b with part c, we get that the two formulas for T_n must be equal. That is, we get the nice formula for the sum of the first n positive integers: $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.)

3. a) If the side of the largest square is F_n, then the side of the next smaller square is F_{n-1}. If we look at the whole rectangle, we see that the height of the rectangle is F_n and the width is $F_n + F_{n-1}$, since the width is made up of a side of the largest square (size F_n) and a side of the next smaller square (size F_{n-1}). But $F_n + F_{n-1}$ is a sum of two consecutive Fibonacci numbers, and that is equal to the next Fibonacci number, which is F_{n+1}. So the width of the rectangle is F_{n+1}, and its area is $F_n \times F_{n+1}$.

b) The area of the rectangle can also be seen as equal to the sum of the areas of all the squares that make up the rectangle. Those squares haves sizes F_1, F_2, F_3, $\ldots$ F_n. The areas of the squares are F_1^2, F_2^2, F_3^2, $\ldots$ F_n^2. The area of the rectangles is the sum of these squares. Since the area is also given by the formula $F_n \times F_{n+1}$, we get that

$$F_n \times F_{n+1} = F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2$$

If we write down what this means for $n = 8$, we get that

$$F_8 \times F_9 = F_1^2 + F_2^2 + F_3^2 + F_4^2 + F_5^2 + F_6^2 + F_7^2 + F_8^2$$

Plugging in the actual Fibonacci numbers (1, 1, 2, 3, 5, 8, 13, 21, 34), this becomes

$$21 \times 34 = 1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 + 13^2 + 21^2$$

or

$$714 = 1 + 1 + 4 + 9 + 25 + 64 + 169 + 441$$

which can be verified on a calculator.

4. a) L_n/L_{n-1} seems to approach the same number as the quotients for the Fibonacci numbers, F_n/F_{n-1}. By $n = 25$, it looks like the first 7 or 8 decimal places of the number are no longer changing: 1.61803398.

b) It seems like no matter what numbers I put in for A_1 and A_2, the values of A_n/A_{n-1} always seem to approach 1.618033988! This is my conjecture.

c) I observe that the values of A_n/A_{n-1} always seem to approach some limiting value. This limiting value changes when I change the multipliers m_1 and m_2, but changing A_1 and A_2 has no effect. This is my conjecture: For the number sequence defined by $A_1 = a$, $A_2 = b$, and $A_n = cA_{n-1} + dA_{n-2}$ for $n > 2$, the values of the quotients A_n/A_{n-1} will always approach some limiting value. Furthermore, changing the initial values A_1 and A_2 has no effect on the limiting value.

5. a) $n = 2$ works since both 2 and 3 are prime.

b) Suppose n is any integer bigger than 2. Then one of n and $n+1$ is an even number bigger than 2. The only even number that is prime is 2 (since all other even numbers are divisible by 2). It follows that n and $n + 1$ can't both be prime, since one of them—the one that is even—is not prime.

6. There are 99 dishonest politicians. There can't be 100 dishonest politicians, since we know that at least one of the politicians is honest. There can't be more than one honest politician either. Suppose there were two. Then, if I talk to those two, **both** of the two that I am talking to would be honest, and this violates condition (b). So, there is exactly one honest politician, and the other 99 are all dishonest.