This lab has a few problems that will be collected at the end of the lab. Answers will be distributed in class tomorrow. If you have extra time during the lab, you can work on the homework assignment.

1. (Not a related rates problem!) Ship A is traveling East at a constant speed of 3 miles per hour. Ship B is traveling North at a constant speed of 4 miles per hour. At 12:00 noon, Ship A is 5 miles due North of Ship B. At what time are the two ships closest together, and what is the distance between them at that time?

2. Let f be a function that is differentiable at $x = a$. The tangent line to $f(x)$ at $x = a$ is a linear function, $L(x)$. This linear function can be written as

$$L(x) = f(a) + f'(a)(x - a)$$

In this formula, a, $f(a)$, and $f'(a)$ are **constants**. (This is covered in Section 4.5; although we are skipping Section 4.5, this formula is something that will be important later, in Calculus II.)

a) Use the formula to find an equation for the tangent line to $f(x) = x^2$ at $x = 2$. (Show your work!)

b) Derive the formula for the general case of the tangent line to $f(x)$ at $x = a$, using the fact that the slope of the tangent line is $f'(a)$. (Show your work and explain your reasoning!)

3. This problem is about *Newton's Method*, a technique that can often be used to find approximate solutions to equations of the form $f(x) = 0$. (The problem is adapted from a suggested project for Chapter 4 of our textbook.)

Let f be a differentiable function. Newton's method starts with some given number x_0, and it produces an infinite sequence of numbers $x_0, x_1, x_2, x_3 \ldots$ that will often converge to a solution of $f(x) = 0$. (In fact, the sequence will approach a solution of $f(x) = 0$ as long as the starting point x_0 is close enough to an actual solution. However, there may be some initial values for which the sequence does not approach a solution.)

Here is how it works: To obtain x_{n+1} from x_n, find the tangent line to $y = f(x)$ at x_n, and let x_{n+1} be the x-value where the tangent line intersects the x-axis.

a) Consider the function $f(x) = x^2 - 2$, and take $x_0 = 2$. Draw a careful graph showing the graph of $y = f(x)$, the tangent line at $x = x_0$, and the point x_1 where the tangent line intersects the x-axis. What is the value of x_1? (By looking at the picture, convince yourself that x_1 is closer than x_0 to a solution of $x^2 - 2 = 0$.)

b) Now, consider the general case of a function $f(x)$ and the sequence of points $x_0, x_1, x_2, \ldots$. Use the formula for the tangent line at $x = x_n$ from the previous problem ($y = f(x_n) + f'(x_n)(x - x_n)$) to show that x_{n+1} is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This is the general formula for Newton's Method. The expectation is that the sequence of numbers defined by this formula will in most cases get closer and closer to a solution of $f(x) = 0$.

c) For the function $f(x) = x^2 - 2$, show that Newton's Method can be written $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$. Use the formula to compute x_1 through x_4, starting from $x_0 = 2$, and check that x_4 is an approximate solution of $x^2 - 2 = 0$. (Use a calculator.)

d) Let $f(x) = x - \cos(x)$. Find a formula for Newton's method for this function. Apply the formula to find x_n for several values of n, starting from $x_0 = 2$. How far do you have to go to get a reasonably good approximation of a solution to $f(x) = 0$?

e) Let $f(x) = \frac{1}{5}x^5 - x + 1$. Apply Newton's method with $x_0 = -1.5$ to find an approximate solution to $f(x) = 0$. What happens if you try to use Newton's method with $x = -1.0$? Why? (This shows that Newton's method doesn't always work!)