Please hand in your responses to the following exercises at the beginning of next Tuesday's lab, September 21. You should work on these problems in a group of three or four students. The questions on this lab are related to material in Sections 2.4 and 2.7. In particular, the last four problems use the "epsilon-delta" definition of limits that is covered in Section 2.7.

1. Use the technique of "rationalizing the numerator" or "rationalizing the denominator" to evaluate each of the following limits.

a) $\displaystyle\lim_{x\to 1} \frac{\sqrt{2-x}-x}{x-1}$ **b)** $\displaystyle\lim_{x\to 0} \frac{x}{\sqrt{3x+9}-\sqrt{x+9}}$

2. Find values of a and b, if possible, so that $\displaystyle\lim_{x\to c} f(x)$ exists for all numbers c, where $f(x)$ is the function $f(x) = \begin{cases} x+2, & \text{if } x<0 \\ ax+b, & \text{if } 0\le x\le 1 \\ x^2-2, & \text{if } x>1 \end{cases}$

3. In this exercise, you will use the Squeeze Theorem to prove that $\displaystyle\lim_{x\to 0} \left(x \sin\left(\tfrac{1}{x}\right)\right) = 0$.

a) Show that $-|x| \le \left(x \sin\left(\frac{1}{x}\right)\right) \le |x|$ for all $x \ne 0$. You will have to consider the cases $x>0$ and $x<0$ separately.

b) Show that $\displaystyle\lim_{x\to 0} |x| = 0$. (Hint: Find the limit from the left and the limit from the right, as x approaches 0.)

c) Apply the Squeeze Theorem.

4. Discuss and explain. Suppose that $f(x)$ and $g(x)$ are functions that are defined for all x. Suppose that $\displaystyle\lim_{x\to 1} g(x) = 2$ and $\displaystyle\lim_{x\to 2} f(x) = 3$. What can you say about of $\displaystyle\lim_{x\to 1}(f\circ g)(x)$? You do not have to give a very rigorous answer.

5. Use the epsilon-delta definition of limits to prove directly that $\displaystyle\lim_{x\to 3}(7x+3) = 21$. (You have to find a specific formula for δ, and show that it works.)

6. Use the epsilon-delta definition of limits to prove directly that $\displaystyle\lim_{x\to 2}(x^2) = 4$. In your proof, use $\delta = \min(1, \frac{\varepsilon}{5})$, so that you have both $\delta \le 1$ and $\delta \le \frac{\varepsilon}{5}$. You will need to use the fact that $|x-2| < \delta$ implies $|x| < 3$, and you should explain why that is true; the explanation can be based on the interpretation of $|x-2|$ as the distance of x from 2.

7. Consider the function $f(x) = \begin{cases} 3 & \text{if } x<1 \\ 2 & \text{if } x>1 \end{cases}$

a) Apply the epsilon-delta definition of limits to show that $\displaystyle\lim_{x\to 1} f(x) \ne 3$. To do this, let $\varepsilon = 1$. Explain why no matter how small δ is chosen, there will be values of x that satisfy $0 < |x-1| < \delta$, but for which $|f(x)-3| \ge \varepsilon$. (And explain why that fact does the trick!) A picture would be helpful.

b) Apply the epsilon-delta definition of limits to show that $\displaystyle\lim_{x\to 1} f(x) \ne 2\tfrac{1}{2}$. To do this, let $\varepsilon = \frac{1}{2}$. Explain why no matter how small δ is chosen, there will be values of x that satisfy $0 < |x-1| < \delta$, but for which $|f(x) - 2\frac{1}{2}| \ge \varepsilon$.

c) Discuss in your own words why there is **no** number L such that $\displaystyle\lim_{x\to 1} f(x) = L$, based on the epsilon-delta definition of limits.

8. Consider the Dirichlet function $g(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$. Using the epsilon-delta definition of limits, explain why $\displaystyle\lim_{x\to a} g(x)$ does not exist for **any** a.