Please hand in your responses to the following exercises at the beginning of the next lab period on October 5. The lab is on continuity and the Intermediate Value Theorem.

You should work on these problems in a group of three or four students. Your group must **discuss each question**, *and each member of the group should understand the question and the group's response. You should* **not** *spend time during lab working on problems on your own. Maybe I will take notes on your interactions during lab, and base part of your lab grade on how you work together...*

1. Is the function $f(x) = \begin{cases} \frac{|x|}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ continuous at 0? (Explain.)

2. Is the function $f(x) = x \cdot |x|$ continuous at 0? (Explain.)

3. Suppose that $f(x) = \begin{cases} x-1, & \text{if } x \leq -1 \\ ax, & \text{if } -1 < x < 1 \\ x^2, & \text{if } x \geq 1 \end{cases}$ Can a value be found for a that makes $f(x)$ continuous at all x? (Explain.)

4. Let $g(x)$ be an odd function ($g(-x) = -g(x)$) which is continuous at $x = 0$.

a) What has to be true about $\lim_{x\to 0^-} g(x)$ and $\lim_{x\to 0^+} g(x)$ because $g(x)$ is continuous at 0?

b) What has to be true about $\lim_{x\to 0^-} g(x)$ and $\lim_{x\to 0^+} g(x)$ because $g(x)$ is odd?

c) What is the value of $g(0)$?

(Explain your answers.)

5. The degree of a polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is n (assuming that $a_n \neq 0$). The Intermediate value theorem can be used to show that every polynomial whose degree is an odd integer has a root. That is, there is at least one number c such that $p(c) = 0$. As an example, consider $p(x) = x^7 + 5x^4 - 3x + 1$.

a) What is $\lim_{x\to\infty} p(x)$? What is $\lim_{x\to-\infty} p(x)$?

b) Deduce that there are numbers a and b such that $p(a) < 0$ and $p(b) > 0$.

c) Use the IVT to show that there is a number c such that $p(c) = 0$.

d) Explain why a similar argument shows that *any* odd-degree polynomial has a root. Draw a picture or two to illustrate your answer.

e) What goes wrong when you try to use the same argument to show that a polynominal of even degree has a root?

6. The Intermediate Value Theorem uses an essential property of the real numbers: The real numbers form a continuum; there is no "hole" at any point along the real number line. Suppose that you only know about rational numbers, numbers that can be written as a quotient of two integers. (In fact, not every number is rational. For example, $\sqrt{2}$ is an irrational number.)

Show that the Intermediate Value Theorem is **false** for the rational numbers by considering the function $f(x) = x^2 - 2$, with domain consisting of the rational numbers only, on the interval $[0, 2]$. After seeing this, are you so sure that the Intermediate Value Theorem is obviously true for the real numbers?

7. Suppose that you get in your car at 1:00 and drive in a straight line to a point exactly 50 miles distant, arriving there at 2:00. Your *average* speed during the trip is 50 miles per hour. Do you believe that there has to be some instant during your journey at which your *instantaneous* speed is exactly 50 miles per hour? Use the Intermediate Value Theorem to argue that there must be at least one such instant. To do this, what do you have to assume about the function that gives your instantaneous velocity as a function of time? Your response to this question should be in the form of an essay.

8. Let $A(w)$ be the function, defined for $w > 1$, such that $A(w)$ is the area of the region bounded by the graph of $y = \frac{1}{x}$ and the lines $y = 0$, $x = 1$, and $x = w$.

- **a)** Draw a picture to illustrate $A(w)$.
- **b)** Show that $A(2) < 1$ by showing that the area represented by $A(2)$ is contained entirely within a rectangle that has area 1.
- **c)** Show that $A(4) > 1$ by finding several rectangles inside the area representing $A(4)$ whose combined area is equal to 1. (You can actually show that $A(3) > 1$, with a little more work. Do that if you want.)
- **d)** Argue that, intuitively, the function $A(w)$ is continuous.
- **e)** Use the Intermediate Value Theorem to deduce that there is a number c between 1 and 4 such that $A(c) = 1$. (This number is actually the mathematical constant e. This definition of e can be used to prove all the properties of e^x and $\ln(x)$, using techniques from Calculus II.)