

*This lab is mostly review for the test on Friday. **Problems 1 through 5 will be collected for grading during the lab at about 2:40, and a solution sheet for the entire lab will be available at that time.** The remaining problems will not be collected or graded.*

1. You need to know the chain rule, and you need to know when and how to apply it. For each of the following derivatives, decide whether the *first step* in computing the derivative is a chain rule problem. If it is (and only if it is), show the result of applying the chain rule. (Show only the first step, and only if it is a chain rule problem.)

$$\begin{array}{lll} \text{a)} \frac{d}{dx} \cos(x^2 e^{-x}) & \text{b)} \frac{d}{dt} t^2 \cos(e^{-t}) & \text{c)} \frac{d}{dx} e^{\sin(\ln(x))} \\ \text{d)} \frac{d}{dx} \sqrt[3]{x^2 - 1} & \text{e)} \frac{d}{dz} \sin(ze^z \cos(z)) & \text{f)} \frac{d}{dz} \sin(ze^z) \cos(z) \end{array}$$

2. Find $\frac{d^2}{dt^2} \left(\frac{1}{1 + e^{-t}} \right)$

3. Suppose that $F(x)$ is a function that satisfies $F'(x) = e^{-x^2}$.

a) Compute $F''(x)$.

b) Define $r(x) = F(3x^2 + 1)$. Compute $r'(x)$, and find the value of $r'(0)$.

4. Suppose that you give something a push to give it a positive initial velocity v_o at time $t = 0$. If there are no forces acting on it (after that initial push), then the acceleration of the object will be zero. Its velocity will be unchanging. And, assuming it starts at position 0, then its position, y , will be given by $y = v_o t$. This object satisfies the differential equation $y'' = 0$.

However, if there is friction, then the frictional force will tend to slow the object down. Frictional force is proportional to velocity and points in the opposite direction. If friction is the only force acting on the object (after the initial push), then the acceleration satisfies the differential equation $y'' = -ky'$ for some positive constant k .

a) Show that the function $y = \frac{v_o}{k}(1 - e^{-kt})$ satisfies the differential equation $y'' = -ky'$.

b) What is $\lim_{t \rightarrow \infty} \frac{v_o}{k}(1 - e^{-kt})$, and what does this say about the *total distance* traveled by the object.

c) Describe the motion of the object, as given by the equation $y = \frac{v_o}{k}(1 - e^{-kt})$, and explain why it makes sense for an object when friction is the only force acting.

5. Is the following function continuous at $x = -2$? at $x = 0$? at $x = 2$? Is it differentiable at $x = -2$? at $x = 0$? at $x = 2$?

$$f(x) = \begin{cases} 4 - x^2, & \text{if } x < -2 \\ 4x, & \text{if } -2 \leq x < 0 \\ x^2, & \text{if } 0 \leq x < 2 \\ 4x - 4, & \text{if } x \geq 2 \end{cases}$$

6. Let $f(x) = \sin^2(x) + \cos^2(x)$. Use the chain rule and other derivative rules to compute $f'(x)$. Simplify the answer, and explain why the answer is so simple.
7. Compute the derivative of $f(x) = 3x^2 + 7x$ directly, using the definition of the derivative as a limit.
8. Each of the following differentiation problems requires several different derivative rules, including the chain rule. Find the derivatives, *showing each step in the computation*.

$$\begin{array}{lll} \text{a)} \frac{d}{dx} \sin(x^2 e^{-x^2}) & \text{b)} \frac{d}{dx} \frac{\sqrt{e^x + 1}}{7x^4 + e^{-x}} & \text{c)} \frac{d}{dt} ((2t + 1)^{10} - (2t - 1)^{11}) \\ \text{d)} \frac{d}{dr} r^2 \cos^2(r^2) & \text{e)} \frac{d}{dx} x e^x \sin^2(x e^x) & \text{f)} \frac{d}{dx} \sqrt{2e^{5x} + \frac{x}{x+1}} \end{array}$$

9. One important physical model is the *harmonic oscillator*. This model describes how something along a line when the force acting on it—and therefore its acceleration—is proportional to the displacement of the object from its equilibrium position, with the direction of the force pulling it back towards its equilibrium position. That is, the displacement y of the object satisfies the differential equation $y'' = -ky$, where k is a positive constant and y is the displacement at time t . Assuming that there is no friction, this is a good model for a pendulum swinging back and forth (if the angle is fairly small) or for a weight bobbing on the end of a spring—or a bungee jumper at the end of a bungee cord.

In a real physical oscillator, the motion will be “damped” by friction, which reduces the size of the oscillation with time. Friction is a force that depends on velocity: the greater the velocity, the greater the force. The direction of the frictional force is in the opposite direction to the velocity. So the force due to friction is of the form $-cy'$, where c is a positive constant. Taking friction into account, the differential equation for a damped harmonic oscillator is $y'' = -ky - cy'$.

- a) Show that the function $y = A \sin(\sqrt{k} \cdot t - B)$ satisfies the differential equation for the harmonic oscillator, where k , A and B are constants, $k > 0$, and $A > 0$. That is, show that $y'' = -ky$.
- b) It is harder to work with the damped harmonic oscillator, but consider the special case $y'' = -2y - 2y'$. Show that the function $y = e^{-t} \sin(t)$ satisfies this equation.
- c) Explain how the function $y = e^{-t} \sin(t)$ makes sense as the equation of a damped harmonic oscillator.