Guided Project 9: Fixed point iteration

Topics and skills: Functions, graphing, calculators

Finding solutions of equations is a common problem in mathematics. Some equations, such as \( x^2 + x - 7 = 0 \), can be solved analytically, while precise solutions to other equations are difficult or impossible to find. To deal with these difficulties, a group of powerful methods, called iterative methods, are used to approximate solutions of equations. Iterative methods are commonly used in science, engineering, and mathematics to make approximations. Root-finders on graphing calculators and computer algebra systems often use iterative methods.

This project introduces a method known as fixed-point iteration, which is used to approximate fixed points of functions. A fixed point of a function \( f \) is a number \( c \) satisfying the equation \( f(x) = x \). If \( c \) is a fixed point of \( f \), then the graphs of \( y = f(x) \) and \( y = x \) intersect at \((c, c)\) as illustrated in Figure 1.

1. Find the fixed points of \( f(x) = x^2 - 2 \) by solving the equation \( x^2 - 2 = x \). Sketch graphs of \( y = f(x) \) and \( y = x \) to illustrate the fixed points. Explain why if you find a fixed point of \( f(x) = x^2 - 2 \), you have also found a root of the equation \( x^2 - 2 - x = 0 \).

2. The exact values of fixed points for most functions cannot be found. For example, a fixed point of \( \cos x \) cannot be found exactly because the equation \( \cos x = x \) cannot be solved analytically. In this case, we use fixed-point iteration to approximate the fixed points of \( f \). Here is how it works.

   Be sure your calculator is in radian mode. Then let \( x_0 = 0.5 \) be an estimate to a fixed point of \( \cos x \).
   a. Now calculate the values \( x_1, x_2, \ldots, x_{10} \), where \( x_1 = \cos x_0 \), \( x_2 = \cos x_1 \), \( x_3 = \cos x_2 \), and so forth.
   b. Notice that the numbers in this list satisfy \( x_{n+1} = \cos x_n \) for \( n = 0, 1, 2, \ldots \). Therefore they get closer and closer to a number that satisfies \( \cos x = x \); that is, they approach a fixed point of \( \cos x \). Based upon your observations in part (a), estimate the value of a fixed point of \( \cos x \), rounded to 4 digits.
   c. Sketch a graph of the functions \( y = \cos x \) and \( y = x \) with a window of \([-2\pi, 2\pi] \times [-2, 2] \). Use these graphs to explain why the fixed point you found in (b) is the only fixed point of the cosine function.
   d. Explain why if you find a fixed point of \( f(x) = \cos x \), you have also found a root of the equation \( \cos x - x = 0 \).

For a given function \( f \) and an initial value \( x_0 \), fixed-point iteration is the process of finding the values of \( x_1, x_2, x_3, \ldots \) defined by \( x_1 = f(x_0) \), \( x_2 = f(x_1) \), \( x_3 = f(x_2) \), \ldots

In general, the value of \( x_{n+1} \) if obtained by calculating \( f(x_n) \), or more concisely, \( x_{n+1} = f(x_n) \) for \( n = 0, 1, 2, \ldots \). If the values \( x_0, x_1, x_2, x_3, \ldots \) approach a single number \( c \), then \( c \) is a fixed point of \( f \).

3. Let \( f(x) = \frac{1}{2} \left( x + \frac{2}{x} \right) \).
   a. Let \( x_0 = 1.5 \). Find and record the values of \( x_1, x_2, x_3, \) and \( x_4 \) using a calculator. Round each answer to eight digits.
   b. Let \( x_0 = -1.5 \). Find and record the values of \( x_1, x_2, x_3, \) and \( x_4 \) using a calculator. Round each answer to eight digits.
   c. Estimate two fixed points of \( f \), rounded to eight digits, based upon the results in parts (a) and (b).
   d. Find the exact values of the fixed points of \( f \). Verify that these results are consistent with your results in parts (b) and (c).
   e. Explain why if you find a fixed point of \( f \), you have also found a root of the equation \( x^2 - 2 = 0 \).