Guided Project 9: Fixed point iteration

Topics and skills: Functions, graphing, calculators

Finding solutions of equations is a common problem in mathematics. Some equations, such as $x^2 + x - 7 = 0$, can be solved analytically, while precise solutions to other equations are difficult or impossible to find. To deal with these difficulties, a group of powerful methods, called **iterative methods**, are used to *approximate* solutions of equations. Iterative methods are commonly used in science, engineering, and mathematics to make approximations. Root-finders on graphing calculators and computer algebra systems often use iterative methods.

This project introduces a method known as **fixed-point iteration**, which is used to approximate *fixed points* of functions. A **fixed point** of a function f is a number c satisfying the equation $f(x) = x$. If c is a fixed point of f, then the graphs of $y = f(x)$ and $y = x$ intersect at (c, c) as illustrated in Figure 1.

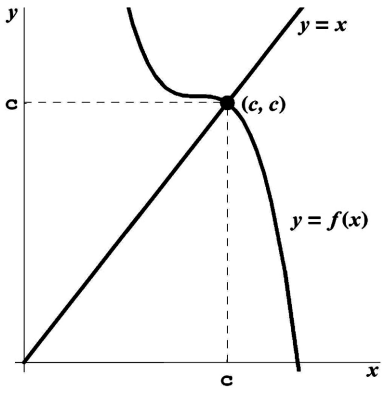

Figure 1

1. Find the fixed points of $f(x) = x^2 - 2$ by solving the equation $x^2 - 2 = x$. Sketch graphs of $y = f(x)$ and $y = x$ to illustrate the fixed points. Explain why if you find a fixed point of $f(x) = x^2 - 2$, you have also found a root of the equation $x^2 - 2 - x = 0$.

2. The exact values of fixed points for most functions cannot be found. For example, a fixed point of $\cos x$ cannot be found exactly because the equation $\cos x = x$ cannot be solved analytically. In this case, we use *fixed-point iteration* to approximate the fixed points of f. Here is how it works.
Be sure your calculator is in radian mode. Then let $x_0 = 0.5$ be an estimate to a fixed point of $\cos x$.
a. Now calculate the values $x_1, x_2, \ldots, x_{10}$, where
$$x_1 = \cos x_0, \quad x_2 = \cos x_1, \quad x_3 = \cos x_2, \text{ and so forth.}$$
Record your results, rounding the results to 4 digits. Describe what you observe in this list of ten numbers.
b. Notice that the numbers in this list satisfy $x_{n+1} = \cos x_n$ for $n = 0, 1, 2, \ldots$ Therefore they get closer and closer to a number that satisfies $\cos x = x$; that is, they approach a fixed point of $\cos x$. Based upon your observations in part (a), estimate the value of a fixed point of $\cos x$, rounded to 4 digits.
c. Sketch a graph of the functions $y = \cos x$ and $y = x$ with a window of $[-2\pi, 2\pi] \times [-2, 2]$. Use these graphs to explain why the fixed point you found in (b) is the only fixed point of the cosine function.
d. Explain why if you find a fixed point of $f(x) = \cos x$, you have also found a root of the equation $\cos x - x = 0$.

For a given function f and an initial value x_0, **fixed-point iteration** is the process of finding the values of $x_1, x_2, x_3, \ldots$ defined by $x_1 = f(x_0)$, $x_2 = f(x_1)$, $x_3 = f(x_2)$, ….
In general, the value of x_{n+1} if obtained by calculating $f(x_n)$, or more concisely, $x_{n+1} = f(x_n)$ for $n = 0, 1, 2, \ldots$. If the values $x_0, x_1, x_2, x_3, \ldots$ approach a single number c, then c is a fixed point of f.

3. Let $f(x) = \frac{1}{2}\left(x + \frac{2}{x}\right)$.
a. Let $x_0 = 1.5$. Find and record the values of x_1, x_2, x_3, and x_4 using a calculator. Round each answer to eight digits.
b. Let $x_0 = -1.5$. Find and record the values of x_1, x_2, x_3, and x_4 using a calculator. Round each answer to eight digits.
c. Estimate two fixed points of f, rounded to eight digits, based upon the results in parts (a) and (b).
d. Find the exact values of the fixed points of f. Verify that these results are consistent with your results in parts (b) and (c).
e. Explain why if you find a fixed point of f, you have also found a root of the equation $x^2 - 2 = 0$.