Calculus II, Spring 2005 — Math 131-01
Lab 13 — April 19, 2005

Please hand in your responses to the following exercises. You should work on these problems in a group and turn in a single solution for the entire group.

1. Compute the sum of each of the following geometric series:

a) $\displaystyle\sum_{n=0}^{\infty} \frac{3}{7 \cdot 5^n}$ **b)** $\displaystyle\sum_{i=3}^{\infty} \frac{2}{3^i}$ **c)** $\displaystyle\sum_{k=1}^{\infty} 0.3^k$

2. In class, we showed that

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \ln(n)$$

We did this using rectangles circumscribed on the curve $y = \frac{1}{x}$. Using *inscribed* rectangles, show that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} < \ln(n)$$

and then deduce that

$$\ln(n) < \sum_{i=1}^{n} \frac{1}{i} < \ln(n) + 1$$

Use this fact to estimate $\displaystyle\sum_{i=1}^{1000} \frac{1}{i}$ and $\displaystyle\sum_{i=1}^{1000000} \frac{1}{i}$.

3. One of Zeno's famous paradoxes goes something like this: Achilles is in a race with a tortoise. Achilles runs at a speed of 10 meters per second, while the tortoise plods along at just 1 meter per second. The tortoise is given a 100 meter head start at the beginning of the race. Zeno argues that Achilles must lose the race because he can never even catch up with the tortoise:

- At 10 seconds, Achilles reaches the tortoise's original position, but by that time, the tortoise has moved forward another 10 meters, so is still in the lead.
- After another 1 second, Achilles has covered the ten meters, but the tortoise has moved forward another meter, so is still in the lead.
- After another 1/10 second, Achilles has covered the meter, but the tortoise has moved forward another 1/10 meter, so is still in the lead.

Clearly, this continues forever. Every time Achilles covers the distance of the tortoise's lead, the tortoise has moved ahead some additional distance and so is still in the lead.

Explain how this paradox can be resolved by using an infinite series. Write down a series that sums up the sequence of distances traveled by Achilles, and use the series to find the exact distance at which Achilles catches up to the tortoise.

4. Example 6 on page 572 of the textbook computes the total distance traveled by a bouncing ball. We do the same in this problem and also consider the question of the length of time for which it bounces. For this problem, suppose that a ball is dropped from a height of 10 feet, and that after each bounce it rises to a height that is exactly 1/2 of the height of the previous bounce. So, the heights of the bounces are 5, $\frac{5}{2}$, $\frac{5}{4}$, $\frac{5}{8}$, $\frac{5}{16}$, and so on. Let D_i be the total distance traveled—up *and* down—during the i^{th} bounce. We are assuming that the ball bounces infinitely many times. But, does this mean that the ball bounces forever?

- **a)** The total distance traveled by the bouncing ball, ignoring the initial drop of 10 feet, is $\sum_{i=1}^{\infty} D_i$. Find the distance by finding the sum of the infinite series.
- **b)** When an object is falling freely—such as *between* bounces—near the surface of the Earth, its height is given by a function of the form $y = -16t^2 + v_o t + y_o$, where time is measured in seconds, height is measured in feet, v_o is the velocity at time zero, and y_o is the height at time zero. For a bounce, the initial height is zero, so the formula is $y = -16t^2 + v_o t$. Based on this, and some Calc I, find the *time* taken by each bounce. (Hint: What is the highest point on the curve $y = -16t^2 + v_o t$?)
- **c)** Write down an infinite series for the total time for which the ball bounces. Find the sum of the series, or show that it diverges to infinity. So, does the ball bounce forever?
- **d)** What do you suppose happens with real balls bouncing in the real world?

5. I first encountered this problem in *Scientific American Magazine* many years ago...

An unfortunate bug, being punished for his sins, is set down at one end of a one-kilometer (1000 meter), infinitely stretchable rubber band. The bug crawls along at one meter per minute, and it must reach the other end of the rubber band. But—and here's the nasty bit—at the end of each minute, the rubber band is stretched so that it grows by another kilometer. At the end of the first minute, it stretches to two kilometers; at the end of the second minute, it stretches to three kilometers; at the end of the third, it stretches to four kilometers; and so on forever. Is the bug's punishment everlasting, or will it eventually reach the other end?

The bug's only hope is the fact that the rubber band stretches evenly, so that the section that is behind the bug stretches proportionately. At the end of the first minute, one meter, or 1/1000, of the total distance is behind the bug, and 999 meters lie ahead. Then the rubber band stretches to two kilometers. Now, two meters are behind the bug, which is still 1/1000 of the total distance, and 1998 meters lie ahead. So, the *fraction* of the total distance traveled does not change when the rubber band stretches.

- **a)** What fraction of the total distance does the bug cover during the second minute? During the third minute? During the n^{th} minute?
- **b)** Write down a sum that represents the *total* fraction that the bug has covered after N minutes.
- **c)** Use what you know about a certain infinite series to show that the bug will, in fact, reach the end eventually.
- **d)** Estimate the time that it takes for the bug to reach the end.