Section 8.5, # 16, 18, 32, 36, 40, 42, 48

Section 8.6, # 40, 42, 44, 46

8.5.16 $\sum_{k=1}^{\infty} \frac{k^6}{k!}$. $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{(k+1)^6}{(k+1)!} \Big/ \frac{k^6}{k!} = \lim_{k\to\infty} \frac{(k+1)^6 \cdot k!}{(k+1)! \cdot k^6}$

$$= \lim_{k\to\infty} \frac{(k)(k-1)\cdots(2)(1) \cdot (k+1)^6}{(k+1)(k)(k-1)(\cdots(2)(1) \cdot k^6} = \lim_{k\to\infty} \frac{(k+1)^6}{(k+1)\cdot k^6}$$

$$= \lim_{k\to\infty} \frac{(k+1)^5}{k^6} = 0$$ (because $\frac{(k+1)^5}{k^6}$ is a rational function where degree of denominator is greater than degree of numerator)

Since $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} < 1$, the series converges by the Ratio Test.

8.5.18 $\sum_{k=1}^{\infty} k^4 2^{-k}$. $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{(k+1)^4 2^{-(k+1)}}{k^4 2^{-k}}$

$$= \lim_{k\to\infty} \left(\frac{(k+1)^4}{k^4}\right) \cdot 2^{-k-1+k} = \lim_{k\to\infty} \frac{(k+1)^4}{k^4} \cdot 2^{-1}$$

$$= 1 \cdot 2^{-1} = \frac{1}{2}.$$ ($\lim_{k\to\infty} \frac{(k+1)^4}{k^4} = 1$, by the rules for limits of rational funtions)

Since $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} < 1$, the series converges by the ratio test.

8.5.32 $\sum_{k=1}^{\infty} \sqrt{\frac{k}{k^3+1}}$. [This looks similar to $\sqrt{\frac{k}{k^3}} = \frac{1}{k}$. Since $\sum \frac{1}{k}$ diverges, I guess that this series does too.]

$k^3+1 \le k^3+k^3$

$k^3+1 \le 2k^3$

$\frac{1}{k^3+1} \ge \frac{1}{2k^3}$

$\frac{k}{k^3+1} \ge \frac{k}{2k^3} = \frac{1}{2k^2}$

$\sqrt{\frac{k}{k^3+1}} \ge \sqrt{\frac{1}{2k^2}} = \frac{1}{\sqrt{2}\,k}$

The series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{2}\,k}$ diverges, since it is a multiple of the divergent series $\sum_{k=1}^{\infty} \frac{1}{k}$.

Since $\sqrt{\frac{k}{k^3+1}} \ge \frac{1}{\sqrt{2}\,k}$ for all k, and $\sum_{k=1}^{\infty} \frac{1}{\sqrt{2}\,k}$ diverges, $\sum_{k=1}^{\infty} \sqrt{\frac{k}{k^3+1}}$ diverges by the Comparison Test.

8.5.36 $\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k+2}}$. [similar to $\sum \frac{1}{k^{3/2}}$, which converges]

$k\sqrt{k+2} > k\sqrt{k} = k^{3/2}$

So $\frac{1}{k\sqrt{k+2}} < \frac{1}{k^{3/2}}$

Since $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ converges, then $\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k+2}}$ also converges, by the Comparison Test.

8.5.40 $\sum_{k=1}^{\infty} \frac{(k!)^3}{(3k)!}$. $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{((k+1)!)^3 \cdot (3k)!}{(3(k+1))! \cdot (k!)^3}$

$$= \lim_{k\to\infty} \left(\frac{(k+1)!}{k!}\right)^3 \cdot \frac{(3k)!}{(3k+3)!}$$

$$= \lim_{k\to\infty} \left(\frac{(k+1)\cdot k(k-1)\cdots 2\cdot 1}{k(k-1)\cdots 2\cdot 1}\right)^3 \left(\frac{3k\cdot(3k-1)(3k-2)\cdots 2\cdot 1}{(3k+3)(3k+2)(3k+1)(3k)\cdots 2\cdot 1}\right)$$

$$= \lim_{k\to\infty} (k+1)^3 \cdot \frac{1}{(3k+3)(3k+2)(3k+1)} = \frac{1}{27}$$

[when multiplied out, it's $\lim_{k\to\infty} \frac{k^3 + \cdots}{27k^3 + \cdots}$]

Since $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \frac{1}{27} < 1$, the series converges by the Ratio Test

8.5.42 $\sum_{k=2}^{\infty} \frac{5\ln(k)}{k}$. $\frac{5\ln(k)}{k} > \frac{1}{k}$. Since $\sum_{k=2}^{\infty} \frac{1}{k}$ diverges, $\sum_{k=2}^{\infty} \frac{5\ln(k)}{k}$ diverges by the Comparison Test

8.5.48 $\sum_{k=2}^{\infty} \frac{1}{k^2 \ln(k)}$. $\ln(k) > 1$ for $k \geq 3$, so $k^2\ln(k) > k^2$ and $\frac{1}{k^2\ln(k)} < \frac{1}{k^2}$ for $k \geq 3$. Since $\sum_{k=2}^{\infty} \frac{1}{k^2}$ converges, so does $\sum_{k=2}^{\infty} \frac{1}{k^2 \ln(k)}$, by the Comparison Test

(8.6.40) $\sum_{k=1}^{\infty}\left(-\frac{1}{3}\right)^k$ is a convergent geometric series.

It converges absolutely since $\sum_{k=1}^{\infty}\left(\frac{1}{3}\right)^k$ is a convergent geometric series

(8.6.42) $\sum_{k=1}^{\infty}\frac{(-1)^k k^2}{\sqrt{k^6+1}}$ Converges Conditionally. It converges by the alternating series test, but $\sum_{k=1}^{\infty}\frac{k^2}{\sqrt{k^6+1}}$ diverges by the comparison test. Compare to $\sum_{k=1}^{\infty}\frac{1}{\sqrt{2}\,k}$.

$\sqrt{k^6+1} \le \sqrt{k^6+k^6} = \sqrt{2}\,k^3$, so $\frac{1}{\sqrt{k^6+1}} \ge \frac{1}{\sqrt{2}\,k^3}$ and

$\frac{k^2}{\sqrt{k^6+1}} \ge \frac{k^2}{\sqrt{2}\,k^3} = \frac{1}{\sqrt{2}\,k}$. Since $\sum_{k=1}^{\infty}\frac{1}{\sqrt{2}\,k}$ diverges, so does $\sum_{k=1}^{\infty}\frac{k^2}{\sqrt{k^6+1}}$

(8.6.44) $\sum_{k=2}^{\infty}\frac{(-1)^k}{\ln(k)}$ Converges conditionally. It converges by the alternating series test, since $\frac{1}{\ln 2} > \frac{1}{\ln 3} > \frac{1}{\ln 4} > \cdots$ and $\lim_{k\to\infty}\frac{1}{\ln k} = 0$. It does not converge absolutely since $\sum_{k=2}^{\infty}\frac{1}{\ln(k)}$ diverges by comparison to the divergent series $\sum_{k=2}^{\infty}\frac{1}{k}$. [$\ln(k) < k$ for $k \ge 2$ so $\frac{1}{\ln(k)} > \frac{1}{k}$]

(8.6.46) $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}e^k}{(k+1)!}$ converges absolutely. $\sum_{k=1}^{\infty}\frac{e^k}{(k+1)!}$ converges by the ratio test:

$$\lim_{k\to\infty}\frac{a_{k+1}}{a_k} = \lim_{k\to\infty}\frac{e^{k+1}\cdot(k+1)!}{(k+2)!\cdot e^k}$$

$$= \lim_{k\to\infty}\frac{e}{k+2} = 0$$

The series converges since this limit is < 1.