Sec. 7.3, # 10, 24
Sec. 7.4, # 8, 10, 12
Sec. 7.7, # 10, 22, 28

7.3.10 $\displaystyle\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\,dx$. Let's do the indefinite integral first...

$\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,dx$ Let $x = 2\sin\theta$, so $dx = 2\cos\theta\,d\theta$, $\theta = \sin^{-1}\left(\frac{x}{2}\right)$

$\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = 2\sqrt{1-\sin^2\theta} = 2\cos\theta$

$$= \int \frac{(2\sin\theta)^2}{2\cos\theta}\cdot 2\cos\theta\,d\theta$$

$$= \int 4\sin^2\theta\,d\theta = 4\cdot\int \frac{1}{2}(1-\cos 2\theta)\,d\theta$$

$$= 2\left(\theta - \frac{1}{2}\sin 2\theta\right) + C$$

$$= 2\theta - \sin(2\theta) + C = 2\sin^{-1}\left(\frac{x}{2}\right) - \sin\left(2\cdot\sin^{-1}\left(\frac{x}{2}\right)\right) + C$$

So

$$\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\,dx = \left[2\sin^{-1}\left(\frac{x}{2}\right) - \sin\left(2\sin^{-1}\left(\frac{x}{2}\right)\right)\Big|_0^{\sqrt{2}}\right]$$

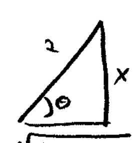
$$= \left(2\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) - \sin\left(2\sin^{-1}\left(\frac{x}{2}\right)\right)\right) - \left(2\sin^{-1}(0) + \sin(2\sin^{-1}(0))\right)$$

$$= \left(2\cdot\frac{\pi}{4} - \sin\left(2\cdot\frac{\pi}{4}\right)\right) - \left(0 + \sin^{-1}(0)\right)$$

$$= \frac{\pi}{2} - \sin\left(\frac{\pi}{2}\right)$$

$$= \underline{\underline{\frac{\pi}{2} - 1}}$$

[We don't need to simplify $2\sin^{-1}\left(\frac{x}{2}\right) - \sin\left(2\cdot\sin^{-1}\left(\frac{x}{2}\right)\right)$ to calculate the definite integral, but we can do so using the identity $\sin(2z) = 2\sin(z)\cos(z)$:



$$2\sin^{-1}\left(\frac{x}{2}\right) - \sin\left(2\sin^{-1}\left(\frac{x}{2}\right)\right)$$
$$= 2\sin^{-1}\left(\frac{x}{2}\right) - 2\sin\left(\sin^{-1}\left(\frac{x}{2}\right)\right)\cos\left(\sin^{-1}\left(\frac{x}{2}\right)\right)$$
$$= 2\sin^{-1}\left(\frac{x}{2}\right) - 2\cdot\frac{x}{2}\cdot\frac{\sqrt{4-x^2}}{2} = \underline{2\sin^{-1}\left(\frac{x}{2}\right) - \frac{x}{2}\sqrt{4-x^2}}.$$
]

7.3.24 $\displaystyle\int \frac{dx}{(81+x^2)^2}$ Let $x = 9\tan\theta$, so $dx = 9\sec^2\theta\, d\theta$,

$(81+x^2) = 81+(9\tan\theta)^2 = 81(1+\tan\theta) = 81\sec^2\theta$

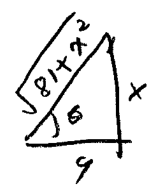
$$= \int \frac{9\sec^2\theta\, d\theta}{(81\sec^2\theta)^2} = \int \frac{1}{729\sec^2\theta}\, d\theta = \frac{1}{729}\int \cos^2\theta\, d\theta$$

$$= \frac{1}{729}\int \frac{1}{2}(1+\cos 2\theta)\, d\theta = \frac{1}{1458}\left(\theta + \frac{1}{2}\sin 2\theta\right)$$

$$= \frac{1}{1458}\left(\tan^{-1}\left(\frac{x}{9}\right) + \frac{1}{2}\cdot 2\sin\left(2\cdot\tan^{-1}\left(\frac{x}{9}\right)\right)\right)$$

$$= \frac{1}{1458}\left(\tan^{-1}\left(\frac{x}{9}\right) + \sin\left(\tan^{-1}\frac{x}{9}\right)\cos\left(\tan^{-1}\left(\frac{x}{9}\right)\right)\right.$$

$$= \frac{1}{1458}\left(\tan^{-1}\left(\frac{x}{9}\right) + \frac{x}{\sqrt{81+x^2}}\cdot\frac{9}{\sqrt{81+x^2}}\right) + C$$



$$= \underline{\underline{\frac{1}{1458}\left(\tan^{-1}\left(\frac{x}{9}\right) + \frac{9x}{81+x^2}\right) + C}}$$

7.4.8 $$\frac{x^2-3x}{x^3-3x^2-4x} = \frac{x^2-3x}{x(x^2-3x-4)} = \frac{x^2-3x}{x(x-4)(x+1)} = \frac{x-3}{x-4(x+1}$$

$$= \underline{\underline{\frac{A}{x} + \frac{B}{x-4} + \frac{C}{x+1}}} \qquad = \underline{\underline{\frac{B}{x-4} + \frac{C}{x+1}}}$$

[A will be 0]

7.4.10 $$\int \frac{8}{(x-2)(x+6)}\, dx = \int \frac{1}{x-2} - \frac{1}{x+6}\, dx = \underline{\underline{\ln|x-2| - \ln|x+6| + C}}$$

$$\frac{8}{(x-2)(x+6)} = \frac{A}{x-2} + \frac{B}{x+6}$$

$8 = A(x+6) + B(x-2)$

$x = -6 \Rightarrow 8 = B(-6-2) \Rightarrow B = -1$

$x = 2 \Rightarrow 8 = A(2+6) \Rightarrow A = 1$

7.4.12 $\displaystyle \int \frac{dt}{t^2-9} = \int \frac{1}{(t-3)(t+3)}\,dt = \int \frac{1/6}{t-3} - \frac{1/6}{t+3}\,dt$

$$= \frac{1}{6}\ln|t-3| - \frac{1}{6}\ln|t+3| + C$$

$$\frac{1}{(t-3)(t+3)} = \frac{A}{t-3} + \frac{B}{t+3}$$

$$1 = A(t+3) + B(t-3)$$

$$t=3 \Rightarrow 1 = A\cdot(3+3) \Rightarrow A = \frac{1}{6}$$

$$t=-3 \Rightarrow 1 = B\cdot(-3-3) \Rightarrow B = -\frac{1}{6}$$

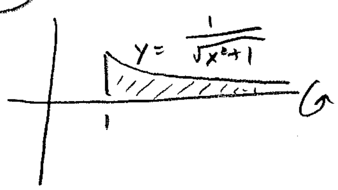
7.7.10 $\displaystyle \int_2^\infty \frac{dx}{x\ln(x)} = \lim_{b\to\infty}\int_2^b \frac{1}{x\ln(x)}\,dx = \lim_{b\to\infty} \ln(\ln x)\Big|_2^b = \lim_{b\to\infty} \ln(\ln b) - \ln(\ln 2)$

$$= \infty$$

$$\left[\int \frac{1}{x\ln(x)}\,dx = \int \frac{1}{z}\,dz = \ln|z| = \ln|\ln(x)| + C \right]$$

$z = \ln(x)$, $dz = \frac{1}{x}\,dx$

7.7.22



$$V = \int_2^\infty \pi\left(\frac{1}{\sqrt{x^2+1}}\right)^2 dx = \int_2^\infty \pi\cdot\frac{1}{x^2+1}\,dx$$

$$= \lim_{b\to\infty} \pi\int_2^b \frac{1}{x^2+1}\,dx = \lim_{b\to\infty} \pi\tan^{-1}(x)\Big|_2^b$$

$$= \lim_{b\to\infty} \pi(\tan^{-1}b - \tan^{-1}) = \pi\left(\frac{\pi}{2} - \tan^{-1}(2)\right)$$

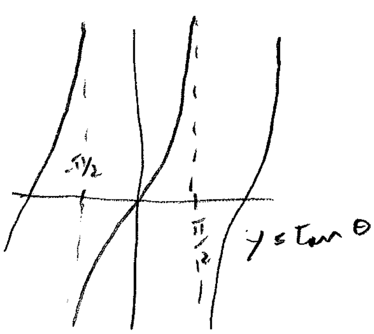
7.7.28 $\displaystyle \int_0^{\pi/2} \tan\theta\,d\theta = \lim_{b\to\pi/2^-}\int_0^b \tan\theta\,d\theta$

$$= \lim_{b\to\frac{\pi}{2}^-}\left[-\ln|\cos\theta|\Big|_0^b\right]$$

[$\tan(\theta)$ is undefined when $\theta = \pi/2$.]

$$= \lim_{b\to\frac{\pi}{2}^-}\left[-\ln|\cos b| - (-\ln|\cos 0|)\right]$$

$$= \lim_{b\to\frac{\pi}{2}^-}\left[-\ln|\cos b| + \ln(1)\right]$$



$$= \infty,$$

since $\cos\frac{\pi}{2} = 0$, so $\ln|\cos b| \to -\infty$ as $b \to \frac{\pi}{2}$.