Math 131-01, Spring 2011 Sample Answers to Lab #1

① a) $\frac{d}{dx}\tan(e^x) = \sec^2(e^x)\cdot\frac{d}{dx}e^x = \sec^2(e^x)\cdot e^x$

b) $$\frac{d}{dt}\left(\frac{e^{2t}-e^{-2t}}{3t^2+\frac{1}{2}t+7}\right) = \frac{(3t^2+\frac{1}{2}t+7)\cdot\frac{d}{dt}(e^{2t}-e^{-2t}) - (e^{2t}-e^{-2t})\frac{d}{dt}(3t^2+\frac{1}{2}t+7)}{(3t^2+\frac{1}{2}t+7)^2}$$

$$= \frac{(3t^2+\frac{1}{2}t+7)\cdot\left(e^{2t}\cdot\frac{d}{dt}(2t) - e^{-2t}\cdot\frac{d}{dt}(-2t)\right) - (e^{2t}-e^{-2t})\cdot(6t+\frac{1}{2})}{(3t^2+\frac{1}{2}t+7)^2}$$

$$= \frac{(3t^2+\frac{1}{2}t+7)\cdot(2e^{2t}+2e^{-2t}) - (e^{2t}-e^{-2t})\cdot(6t+\frac{1}{2})}{(3t^2+\frac{1}{2}t+7)^2}$$

c) $$\frac{d}{dx}\left(x\sin(x^{17})\right)^{17} = 17\left(x\sin(x^{17})\right)^{16}\cdot\frac{d}{dx}\left(x\sin(x^{17})\right)$$

$$= 17\cdot\left(x\sin(x^{17})\right)^{16}\cdot\left[x\frac{d}{dx}\sin(x^{17}) + \sin(x^{17})\cdot\frac{d}{dx}x\right]$$

$$= 17\left(x\sin(x^{17})\right)^{16}\cdot\left(x\cos(x^{17})\cdot\frac{d}{dx}(x^{17}) + \sin(x^{17})\cdot 1\right)$$

$$= 17\left(x\sin(x^{17})\right)^{16}\cdot\left(\left(x\cos(x^{17})\cdot 17x^{16}\right) + \sin(x^{17})\right)$$

② a) $$\int x^{1/2}\,dx = \frac{x^{1/2+1}}{1/2+1} + C = \frac{x^{3/2}}{3/2} + C = \frac{2}{3}x^{3/2} + C$$

b) $$\int \sin(t) - \cos(t)\,dt = -\cos(t) - \sin(t) + C$$

c) $$\int\frac{1+x}{x\sqrt{x}}\,dx = \int\frac{1+x}{x^{3/2}}\,dx = \int x^{-3/2} + x^{-1/2} + C$$

$$= \frac{x^{-3/2+1}}{-3/2+1} + \frac{x^{-1/2+1}}{-1/2+1} + C = -2x^{-1/2} + 2x^{1/2} + C$$

d) $$\int z(1+z)^2\,dz = \int z(1+2z+z^2)\,dz$$

$$= \int z + 2z^2 + z^3\,dz = \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{4}z^4 + C$$

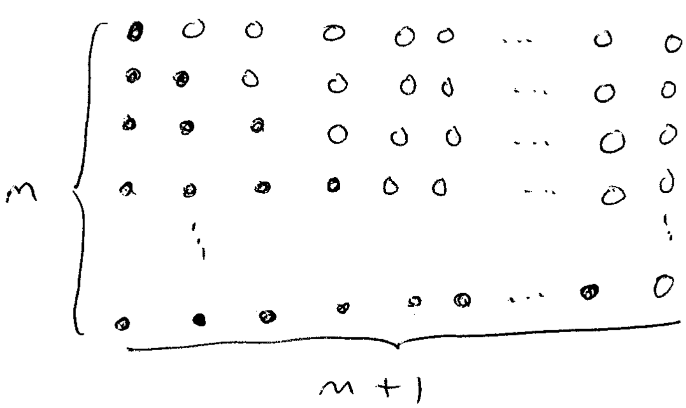
(3) a) There are 7 rows and 8 columns of circles, for a Total of $7\cdot(7+1)$ circles. Half of The circles are gray, so There are $\frac{7\cdot(7+1)}{2}$ gray circles.

We can also count The gray circles as follows: 1 in The 1^{st} row + 2 in The 2^{nd} row

+ 3 in The 3^{rd} row + ⋯ + 7 in The 7^{th} row,

giving $1+2+3+\cdots+7$ as The number of gray circles. When we count The same Thing in Two different ways, we have to get The same number, so $1+2+3+\cdots+7 = \frac{7\cdot(7+1)}{2}$,

b) It's pretty clear That The same pattern holds for all m.



The number of filled-in circles is half The Total number, or $\frac{m(m+1)}{2}$.

Counting by rows, it is also $1+2+\cdots+m$,

So $1+2+\cdots+m = \frac{m(m+1)}{2}$

c) Dividing $[0,1]$ into m subintervals, The length of each subinterval is $\frac{1}{m}$, and The division points are $0, \frac{1}{m}, \frac{2}{m}, \frac{3}{m}, \ldots, \frac{m}{m}$. The Right Riemann Sum Then becomes $f(\frac{1}{n})\cdot\frac{1}{n} + f(\frac{2}{n})\cdot\frac{1}{n} + \cdots + f(\frac{n}{n})\cdot\frac{1}{n}$

$$= \left(\frac{1}{2}\cdot\frac{1}{n}\right)\cdot\frac{1}{n} + \left(\frac{1}{2}\cdot\frac{2}{n}\right)\cdot\frac{1}{n} + \left(\frac{1}{2}\cdot\frac{3}{n}\right)\cdot\frac{1}{n} + \cdots + \left(\frac{1}{2}\cdot\frac{n}{n}\right)\cdot\frac{1}{n}$$

$$= \frac{1}{2n^2}\left(1+2+3+\cdots+n\right)$$

The width of the k^{th} rectangle is $\frac{1}{n}$ and the height is $f(\frac{k}{n})$, or $\frac{1}{2}\cdot\frac{k}{n}$. So the area of the k^{th} rectangle is $\left(\frac{1}{2}\cdot\frac{k}{n}\right)\cdot\frac{1}{n}$. To get the Riemann Sum, we add up the areas of all the n rectangles, giving the sum above.

d) Using the formula from b), we can write

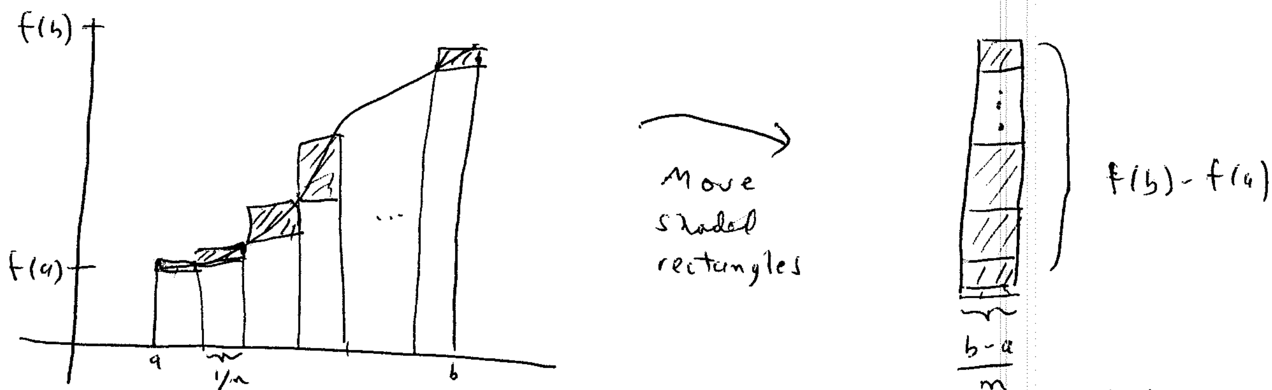
$$\frac{1}{2n^2}\left(1+2+3+\cdots+n\right) = \frac{1}{2n^2}\cdot\left(\frac{n(n+1)}{2}\right) = \frac{n+1}{4n}.$$

Letting $n \to \infty$, $\lim_{n\to\infty} \frac{n+1}{4n} = \frac{1}{4}$. The area under the line is a triangle with base 1 and with height $\frac{1}{2}$. The area of the triangle is $\frac{1}{2}\cdot 1\cdot\frac{1}{2} = \frac{1}{4}$. This is the same as the limit of the Riemann sums as $n \to \infty$.

This makes sense because the error in the Riemann Sum is represented by the areas of the small triangles above the line. As $n \to \infty$, these triangles get smaller and the error goes to zero.

(4) a) The ~~left~~ area represented by the left Riemann sum lies entirely <u>inside</u> the area under the curve, so the Rieman sum is <u>less</u> than the area. The area represented by the Right Riemann sum entirely <u>contains</u> the area under the curve, and so is bigger.

b) An exact formula for $R_n - L_n$ is $(f(b)-f(a))\cdot\frac{(b-a)}{n}$.
One way to see this:



The difference $R_n - L_n$ consists of the areas of the shaded rects.

They can be rearranged into one rectangle with a height of $f(b)-f(a)$ and width $\frac{b-a}{n}$

c) $\lim_{n\to\infty}(R_n - L_n) = \lim_{n\to\infty}\left((f(b)-f(a))\left(\frac{b-a}{n}\right)\right)$

$$= (f(b)-f(a))\cdot(b-a)\cdot\lim_{n\to\infty}\frac{1}{n} = \underline{0}$$

d) Let A be the area under the curve. L_n is an underestimate for A and R_n is an overestimate. That is, $L_n < A < R_n$. As $n\to\infty$, $R_n - L_n \to 0$, so $\lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n$. Since L_n is always less than A and R_n is always greater than A, and L_n and R_n get closer and closer to each other, they must both approach A: