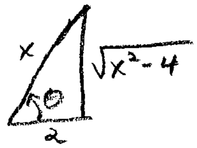
(1)



$\sec\theta = \frac{x}{2}$ $\quad\sin\theta = \frac{\sqrt{x^2-4}}{x}$ $\quad\tan\theta = \frac{\sqrt{x^2-4}}{2}$

$\cos\theta = \frac{1}{\sec\theta} = \frac{2}{x}$ $\quad\csc\theta = \frac{x}{\sqrt{x^2-4}}$ $\quad\cot\theta = \frac{2}{\sqrt{x^2-4}}$

(2) a) $\int \frac{1}{1-z^2}\,dz = \int \frac{1}{(1-z)(1+z)}\,dz$

$$\frac{1}{(1-z)(1+z)} = \frac{A}{1-z} + \frac{B}{1+z} \Rightarrow 1 = A(1+z) + B(1-z)$$

$$z=1 \Rightarrow 1 = A(1+1) \Rightarrow A = \frac{1}{2}$$

$$z=-1 \Rightarrow 1 = B(1-(-1)) = B\cdot 2 \Rightarrow B = \frac{1}{2}$$

So $\int \frac{1}{(1-z)(1+z)}\,dz = \int \frac{1/2}{1-z} + \frac{1/2}{1+z}\,dz$

$$= -\frac{1}{2}\ln|1-z| + \frac{1}{2}\ln|1+z| + C$$

$$= \frac{1}{2}\left(\ln|1+z| - \ln|1-z|\right) + C$$

b) $\int \frac{1}{1-z^2}\,dz = \int \frac{1}{\cos^2\theta}\cdot\cos\theta\,d\theta = \int \frac{1}{\cos\theta}\,d\theta$

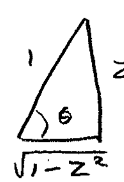
Let $z = \sin\theta$

$dz = \cos\theta\,d\theta$

$1-z^2 = 1-\sin^2\theta = \cos^2\theta$

$$= \int \sec\theta\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$



$\tan\theta = \frac{z}{\sqrt{1-z^2}}$

$\cos\theta = \sqrt{1-z^2}$

$\sec\theta = \frac{1}{\sqrt{1-z^2}}$

$$= \ln\left|\frac{1}{\sqrt{1-z^2}} + \frac{z}{\sqrt{1-z^2}}\right| + C$$

c) $\ln\left|\frac{1}{\sqrt{1-z^2}} + \frac{z}{\sqrt{1-z^2}}\right| = \ln\left|\frac{1+z}{\sqrt{1-z^2}}\right|$

$$= \ln|1+z| - \ln|\sqrt{1-z^2}| = \ln|1+z| - \frac{1}{2}\ln|1-z^2|$$

$$= \ln|1+z| - \frac{1}{2}\left(\ln|(1+z)(1-z)|\right)$$

$$= \ln|1+z| - \frac{1}{2}\left[\ln|1+z| + \ln|1-z|\right] = \frac{1}{2}\left[\ln|1+z| - \ln|1-z|\right]$$

(3) $$\frac{x+2}{x(2x-1)(x+1)^2} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+1} + \frac{D}{(x+1)^2}$$

$$x-2 = A(2x-1)(x+1)^2 + Bx(x+1)^2 + Cx(2x-1)(x+1) + Dx(2x-1)$$

$$x=0 \Rightarrow -2 = A(-1)(1) \Rightarrow -2 = -1\cdot A \Rightarrow A = 2$$

$$x=-1 \Rightarrow -3 = D(-1)(-3) \Rightarrow -3 = 3D \Rightarrow D = -1$$

$$x=\tfrac{1}{2} \Rightarrow -\tfrac{3}{2} = B\left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right)^2 \Rightarrow -\tfrac{3}{2} = \tfrac{9}{8}B \Rightarrow B = -\tfrac{4}{3}$$

$$x=1 \Rightarrow -1 = A(1)(2)^2 + B(1)(2)^2 + C(1)(1)(2) + D(1)(1)$$

$$-1 = 4A + 4B + 2C + D$$

$$-1 = 4\cdot 2 + 4\left(-\tfrac{4}{3}\right) + 2C + (-1) = \tfrac{5}{3} + 2C$$

$$-8/3 = 2C$$

$$-4/3 = C$$

So $$\int \frac{x-2}{x(2x-1)(x+1)^2}\,dx = \int \frac{2}{x} + \frac{-4/3}{2x-1} + \frac{-4/3}{x+1} + \frac{-1}{(x+1)^2}\,dx$$

$$= 2\ln|x| - \frac{4}{3}\cdot\frac{1}{2}\ln|2x-1| - \frac{4}{3}\ln|x+1| + \frac{1}{x+1} + C$$

$$= 2\ln|x| - \frac{2}{3}\ln|2x-1| - \frac{4}{3}\ln|x+1| + \frac{1}{x+1} + C$$

(4) $$\int \frac{1}{\sqrt{x}\sqrt{4+x}}\,dx$$

Let $z = \sqrt{x}$ $\quad x = z^2$

$dz = \frac{1}{2\sqrt{x}}\,dx$

$dx = 2\sqrt{x}\,dz = 2z\,dz$

$$= \int \frac{1}{z\sqrt{4+z^2}}\cdot 2z\,dz$$

$$= 2\int \frac{1}{\sqrt{4+z^2}}\,dz$$

Let $z = 2\tan\theta$, $\theta = \tan^{-1}\left(\frac{z}{2}\right)$

$dz = 2\sec^2\theta\,d\theta$

$\sqrt{4+z^2} = \sqrt{4+4\tan^2\theta} = 2\sqrt{1+\tan^2\theta}$

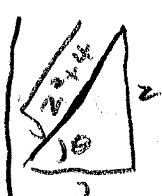
$= 2\sqrt{\sec^2\theta} = 2\sec\theta$

$$= 2\int \frac{1}{2\sec\theta}\cdot 2\sec^2\theta\,d\theta$$

$$= 2\int \sec\theta\,d\theta$$

$$= 2\ln|\sec\theta + \tan\theta| + C$$

$$= 2\ln\left|\frac{\sqrt{z^2+4}}{2} + \frac{z}{2}\right| + C = 2\ln\left|\frac{\sqrt{x+4}}{2} + \frac{\sqrt{x}}{2}\right| + C$$



$\tan\theta = \frac{z}{2}$

$\sec\theta = \frac{\sqrt{z^2+4}}{2}$

(5) $\int \frac{e^x - 1}{e^{2x} - 5e^x + 6}\, dx$ Let $z = e^x$

$dz = e^x dx$

$dx = \frac{1}{e^x} \cdot dz = \frac{1}{z} dz$

$$= \int \frac{z-1}{z^2 - 5z + 6} \cdot \frac{1}{z}\, dz$$

$$= \int \frac{z-1}{z(z-2)(z-3)}\, dz = \int \frac{-1/6}{z} + \frac{-1/2}{z-2} + \frac{2/3}{z-3}\, dz$$

$$= -\frac{1}{6} \ln|z| - \frac{1}{2} \ln|z-2| + \frac{2}{3} \ln|z-3| + C$$

$$= -\frac{1}{6} x - \frac{1}{2} \ln|e^x - 2| + \frac{2}{3} \ln|e^x - 3| + C$$

$$\frac{z-1}{z(z-2)(z-3)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z-3}$$

$z - 1 = A(z-2)(z-3) + Bz(z-3) + Cz(z-2)$

$z = 0 \Rightarrow -1 = A(-2)(-3) \Rightarrow A = -1/6$

$z = 2 \Rightarrow 1 = B \cdot 2 \cdot (-1) \Rightarrow B = -1/2$

$z = 3 \Rightarrow 2 = C \cdot 3 \cdot (1) \Rightarrow C = 2/3$

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(6) $\int_1^\infty \frac{1}{x^p}\, dx = \int_1^\infty x^{-p}\, dx = \lim_{b \to \infty} \int_1^b x^{-p}\, dx = \lim_{b \to 0} \left. \frac{x^{-p+1}}{-p+1} \right|_1^b$

$= \lim_{b \to \infty} \frac{1}{1-p}\left(b^{1-p} - 1\right)$, assuming $p \neq 1$

For $p > 1$, $1 - p < 0$, and $b^{1-p} \to 0$ as $p \to \infty$,

so for $p > 1$, $\int_1^\infty \frac{1}{x^p}\, dx = \frac{1}{1-p} \cdot (0 - 1) = \boxed{\frac{1}{p-1}}$

For $0 < p < 1$, $1 - p > 0$, and $b^{1-p} \to \infty$ as $p \to \infty$

so for $0 < p < 1$, $\int_1^\infty \frac{1}{x^p}\, dx = \infty$

For $p = 1$, we long ago calculated that $\int_1^\infty \frac{1}{x}\, dx = \infty$.