Math 131, Spring 2011 Sample Answers to Lab #11

① a) $\left\{\frac{3^n}{7^n}\right\}_{n=1}^{\infty}$. $\frac{3^n}{7^n} = \left(\frac{3}{7}\right)^n$. This is a geometric sequence that converges to 0.

b) $\left\{(-1)^n \frac{n}{n+3}\right\}_{n=0}^{\infty} = \left\{0, -\frac{1}{4}, \frac{2}{5}, -\frac{3}{6}, \frac{4}{7}, -\frac{5}{8}, \frac{6}{9}, -\frac{7}{10}, \dots\right\}$

Note that $\frac{n}{n+3} \to 1$ as $n \to \infty$, but the signs of the terms alternate between + and −. So, even-numbered terms are approaching 1 while odd-numbered terms approach −1. This sequence diverges (by "oscillation")

c) $\left\{\frac{2k+1}{1000000}\right\}_{k=1}^{\infty}$. $\lim_{x\to\infty} \frac{2x+1}{1000000} = \frac{1}{1000000} \cdot \lim_{x\to\infty}(2x+1) = \infty$.

The constant $\frac{1}{1000000}$ makes no difference to convergence.

d) $\left\{2 + \frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$ $2 - \frac{1}{n} \le 2 + \frac{(-1)^n}{n} \le 2 + \frac{1}{n}$.

$\lim_{n\to\infty} 2 - \frac{1}{n} = \lim_{n\to\infty} 2 + \frac{1}{n} = 2$, so $\left\{2 + \frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$

converges to 2 by the squeeze theorem.

② a) $\int x^3 e^{x^2}\,dx$ $\quad z = x^2$, $dz = 2x\,dx$

$= \frac{1}{2}\int z e^z\,dz$ $\quad u = z \quad dv = e^z\,dz$, $du = dz \quad v = e^z$

$= \frac{1}{2}\left(ze^z - \int e^z\,dz\right) + C$

$= \frac{1}{2}\left(ze^z - e^z\right) + C$

$= \frac{1}{2}\left(x^2 e^{x^2} - e^{x^2}\right) + C$

OR $\quad u = x^2 \quad dv = xe^{x^2}\,dx$, $du = 2x\,dx \quad v = \frac{1}{2}e^{x^2}$

$\int x^3 e^{x^2}\,dx = \int x^2 \cdot xe^{x^2}\,dx$

$= x^2 \cdot \frac{1}{2}e^{x^2} - \int \frac{1}{2}e^{x^2} \cdot 2x\,dx$

$= \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}\int e^{x^2} \cdot 2x\,dx$

$= \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + C$

b) $\int x^3 \sin(x^4)\,dx$ $\quad u = x^4$, $du = 4x^3\,dx$

[Simple substitution!]

$= \frac{1}{4}\int \sin(u)\,du$

$= -\frac{1}{4}\cos(u) + C$

$= -\frac{1}{4}\cos(x^4) + C$

c) $\int \frac{\sin^3(\theta)}{\cos^2(\theta)}\,d\theta = \int \frac{\sin^2(\theta)}{\cos^2(\theta)} \cdot \sin(\theta)\,d\theta$

$= \int \frac{1-\cos^2(\theta)}{\cos^2(\theta)} \cdot \sin(\theta)\,d\theta$ $\qquad w = \cos(\theta)$, $dw = -\sin(\theta)\,d\theta$

$= -\int \frac{1-w^2}{w^2}\,dw$

$= -\int \frac{1}{w^2} - 1\,dw$

$= -\left(-\frac{1}{w} - w\right) + C$

$= \frac{1}{w} + w + C = \frac{1}{\cos(\theta)} + \cos(\theta) + C = \underline{\underline{\sec(\theta) + \cos(\theta) + C}}$

(3) $\int x e^{-2x}\,dx$ $\qquad w = x$, $dw = dx$ $\qquad dv = e^{-2x}\,dx$, $v = -\frac{1}{2}e^{-2x}$

$= -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}\,dx = -\frac{1}{2}xe^{-2x} + \frac{1}{2}\cdot\left(-\frac{1}{2}e^{-2x}\right) + C$

$= -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$

So $\int_0^\infty xe^{-2x}\,dx = \lim_{b\to\infty} \int_0^b xe^{-2x}\,dx$

$= \lim_{b\to\infty}\left[-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\Big|_0^b\right]$

$= \lim_{b\to\infty}\left[\left(-\frac{1}{2}be^{-2b} - \frac{1}{4}e^{-2b}\right) - \left(0 - \frac{1}{4}e^0\right)\right]$

$= \left(\lim_{b\to\infty}\left(-\frac{1}{2}be^{-2b}\right)\right) - 0 - 0 + \frac{1}{4} = \underline{\underline{\frac{1}{4}}}$

__Note:__ $\lim_{b\to\infty} -\frac{1}{2}be^{-2b} = -\frac{1}{2}\lim_{b\to\infty}\frac{b}{e^{2b}} = -\frac{1}{2}\lim_{b\to\infty}\frac{1}{2e^{2b}} = 0$

L'Hôpital's Rule

since $e^{2b} \to \infty$ as $b \to \infty$

(4) $\{a_k\}_{k=1}^{\infty}$ is monotonic and bounded. [It is bounded by a_1.] So it must converge. Since $a_k > 0$ for all k, They can't approach a negative number. [For $L>0$, The distance between a_k and $-L$ is always greater Than L, so a_k can't approach L.] So The limit must be ≥ 0.

The sequence $\{\frac{1}{m}\}_{m=1}^{\infty}$ is a decreasing sequence of positive Terms That converges to 0.

The sequence $\{1+\frac{1}{m}\}_{m=1}^{\infty}$ is a decreasing sequence of positive Terms That converges to 1.

(5) $\dfrac{x-1}{(3x-1)(x+2)} = \dfrac{A}{3x-1} + \dfrac{B}{x+2}$

$x-1 = A(x+2) + B(3x-1)$

$x=\frac{1}{3} \Rightarrow \frac{1}{3}-1 = A(\frac{1}{3}+2)$

$-\frac{2}{3} = \frac{7}{3}A$

$A = -\frac{2}{3}\cdot\frac{3}{7} = -\frac{2}{7}$

$x=-2 \Rightarrow -2-1 = B(3\cdot(-2)-1)$

$-3 = B\cdot(-7)$

$B = 3/7$

So: $\displaystyle\int \frac{x-1}{(3x-1)(x+2)} = \int \frac{-\frac{2}{7}}{3x-1} + \frac{\frac{3}{7}}{x+2}\,dx$

$= -\frac{2}{7}\cdot\frac{1}{3}\ln|3x-1| + \frac{3}{7}\ln|x+2| + C$

$= \underline{-\frac{2}{21}\ln|3x-1| + \frac{3}{7}\ln|x+2| + C}$

(6) $\displaystyle\int \frac{8}{(x^2+16)^{3/2}}\,dx$

Let $x = 4\tan\theta$, $dx = 4\sec^2\theta\,d\theta$

$x^2+16 = (4\tan\theta)^2+16 = 16\tan^2\theta+16$

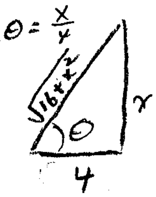
$= 16(\tan^2\theta+1) = 16\sec^2\theta$

$= \displaystyle\int \frac{8}{(16\sec^2\theta)^{3/2}}\cdot 4\sec^2\theta\,d\theta$

$= \displaystyle\int \frac{32}{16^{3/2}\cdot\sec^3\theta}\cdot\sec^2\theta\,d\theta = \frac{32}{64}\int\frac{1}{\sec\theta}\,d\theta$

$= \frac{1}{2}\displaystyle\int\cos\theta\,d\theta = \frac{1}{2}\sin\theta + C = \underline{\frac{x}{2\sqrt{16+x^2}} + C}$

$\tan\theta = \frac{x}{4}$



$\sin\theta = \dfrac{x}{\sqrt{16+x^2}}$

(7) This is radioactive decay, $y(t) = Ae^{-kt}$. The half-life is $\frac{\ln(2)}{k}$, so $\frac{\ln(2)}{k} = 8.0197$, and $k = \frac{\ln(2)}{8.0197}$. We can take A, the initial amount, to be 100 (percent) and we want to find the time when the amount is 1 (percent), so solve

$$1 = 100e^{-\left(\frac{\ln(2)}{8.0197}t\right)} \text{ for } t:$$

$$\frac{1}{100} = e^{-\frac{\ln(2)}{8.0197}t}$$

$$\ln\left(\frac{1}{100}\right) = -\frac{\ln(2)}{8.0197}t$$

$$t = \frac{\ln\left(\frac{1}{100}\right)}{-\frac{\ln(2)}{8.0197}} = 8.0197 \cdot \left(\frac{\ln(0.01)}{-\ln(2)}\right) \approx \underline{53.2817 \text{ days}}$$

(8) A differential equation is an equation that involves the derivatives of some (unknown) function. A function $f(x)$ is a solution of the differential equation if plugging in $f(x)$, $f'(x)$, ... for the unknown function and its derivatives gives a true equation.

(9) $\frac{dy}{dx} = -4x^3y^2$. Use separation of variables.

$$\frac{1}{y^2}\,dy = -4x^3\,dx$$

$$\int \frac{1}{y^2}\,dy = \int -4x^3\,dx$$

$$-\frac{1}{y} = -x^4 + C_1$$

$$\frac{1}{y} = x^4 - C_1 = x^4 + C$$

$$1 = y(x^4 + C)$$

$$y = \frac{1}{x^4 + C}$$

The general solution is
$$y = \frac{1}{x^4 + C}$$

Given the initial value $y(0) = \frac{1}{2}$, we can solve for C:

$$\frac{1}{2} = \frac{1}{0^4 + C} = \frac{1}{C}$$

$$C = 2$$

So the solution to the initial value problem is

$$\underline{y = \frac{1}{x^4 + 2}}$$