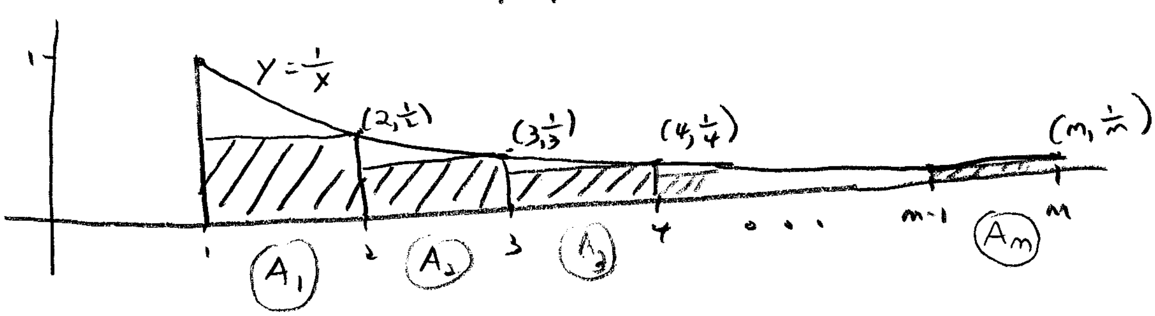
Math 131, Spring 2011 Sample Answers to Lab # 12

① Zeno shows us an infinite number of events and asserts that such a sequence goes on forever and that therefore you can never reach its end. But from the modern point of view, we can add up all the times between the events, and we can show that the sum is finite. Thus, although there is an infinite sequence of events, it is over in a finite time. In fact, the time is given by $10+1+\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\cdots$. This is a geometric series with $a=10$, $r=1/10$, and sum $\frac{a}{1-r}=\frac{10}{1-\frac{1}{10}}=\frac{10}{\frac{9}{10}}=10\cdot\frac{10}{9}=\frac{100}{9}$.

After $\frac{100}{9}$ seconds, Achilles catches up to the Tortoise. At that time, going 10 meters per second, Achiles has travelled $\frac{1000}{9}$ meters.

The distance can also be computed directly as the sum of the geometric series $100+10+1+\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\cdots$

② We note that $\ln(m)=\int_1^m \frac{1}{x}\,dx$. Consider this picture:



The area under the curve is $\int_1^m \frac{1}{x}\,dx$, or $\ln(m)$. The area of the shaded rectangles, $A_1+A_2+\cdots+A_m$, is $\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}$.

Since the rectangles are completely contained in the region under the curve, we see that

$$\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}<\int_1^m \frac{1}{x}\,dx=\ln(m)$$

Adding 1 to this sum gives $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}<1+\ln(m)$.

So $\ln(m+1)<\sum_{i=1}^{m}\frac{1}{i}<\ln(m)$. In particular,

$$\ln(1001)<\sum_{i=1}^{1000}\frac{1}{i}<1+\ln(1000), \quad \text{or} \quad 6.909<\sum_{i=1}^{1000}\frac{1}{i}<7.908$$

and

$$\ln(1{,}000{,}001)<\sum_{i=1}^{1000000}\frac{1}{i}<1+\ln(1000000), \quad \text{or} \quad 13.816<\sum_{i=1}^{1000000}\frac{1}{i}<14.816.$$

(3) a) At the start of the m^{th} minute, the rubber band is m yards, or 36m inches. During the m^{th} minute, Sisyphus travels 1 inch, or $\frac{1}{36m}$ of the total distance.

b) So the total fraction of the distance traveled after m minutes is $\frac{1}{36}+\frac{1}{36\cdot 2}+\frac{1}{36\cdot 3}+\cdots+\frac{1}{36\cdot m}=\frac{1}{36}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}\right)$

c) Sisyphus will reach the end when the total fraction that he has covered exceeds 1, that is when $\frac{1}{36}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}\right)>1$, or $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}>36$. Since $\sum_{k=1}^{\infty}\frac{1}{m}$ diverges to ∞, the partial sum $\sum_{k=1}^{m}\frac{1}{k}$ will eventually exceed 36.

d) $\sum_{k=1}^{m}\frac{1}{k}>\ln(m+1)$, so $\sum_{k=1}^{m}\frac{1}{k}$ will be >36 if $\ln(m+1)>36$, that is when $m+1>e^{36}$, or $m>e^{36}+1$, which is about 4.3×10^{15} minutes or 8,200,000,000 years.

e) Since $\sum_{k=1}^{\infty}\frac{1}{k}$ diverges to infinity, the punishment will end no matter how long the rubber band is initially. Tantalus's punishment ends after m minutes, where $\sum_{k=1}^{m}\frac{1}{k}>21{,}120$. That is after about $m=e^{21120}$ minutes, or about 10^{9172} minutes.

(4) a) $D_1=10$, $D_2=5$, $D_3=\frac{5}{2}$, ... so $\sum_{i=1}^{\infty}D_i=10+5+\frac{5}{2}+\frac{5}{4}+\cdots$. This is a geometric series with $a=10$ and $r=\frac{1}{2}$. The sum is $\frac{10}{1-\frac{1}{2}}$, or 20. So the ball travels a total of 20 feet as it bounces infinitely often.

b) When the ball hits the ground, $y=0$. That is, $0=-16t^2+V_0t=t(-16t+V_0)$. The solutions are $t=0$, $t=\frac{V_0}{16}$. $t=0$ is when the bounce starts, $t=\frac{V_0}{16}$ is when it ends, so the duration of the bounce is $\frac{V_0}{16}$. The highest point occurs when $y'=0$, or $-32t+V_0=0$, or $t=\frac{V_0}{32}$. That height is $y\left(\frac{V_0}{32}\right)=-16\left(\frac{V_0}{32}\right)^2+V_0\left(\frac{V_0}{32}\right)=\frac{V_0^2}{64}$. The height of the i^{th} bounce is $\frac{5}{2^i}$, so $\frac{V_0^2}{64}=\frac{5}{2^i}$ and $V_0=\sqrt{\frac{5\cdot 64}{2^i}}=\frac{8\sqrt{5}}{2^{i/2}}$ and the time for i^{th} bounce is $\frac{V_0}{16}=\frac{\sqrt{5}}{2}\cdot\frac{1}{(\sqrt{2})^i}$.

c) Total time is $\sum_{i=0}^{\infty}\frac{\sqrt{5}}{2}\cdot\frac{1}{(\sqrt{2})^i}=\frac{\sqrt{5}/2}{1-1/\sqrt{2}}\approx 3.8172$ seconds. $\left[\text{geometric series, } a=\frac{\sqrt{5}}{2},\ r=\frac{1}{\sqrt{2}}\right]$