You should turn in all questions from this lab next week at the April 21 lab.
For this lab, you are required write out your answers in essay form.

1. One of Zeno's famous paradoxes goes something like this: Achilles is in a race with a tortoise. Achilles runs at a speed of 10 meters per second, while the tortoise plods along at just 1 meter per second. The tortoise is given a 100 meter head start at the beginning of the race. Zeno argues that Achilles must lose the race because he can never even catch up with the tortoise:

- At 10 seconds, Achilles reaches the tortoise's original position, but by that time, the tortoise has moved forward another 10 meters, so is still in the lead.
- After another 1 second, Achilles has covered the ten meters, but the tortoise has moved forward another meter, so is still in the lead.
- After another 1/10 second, Achilles has covered the meter, but the tortoise has moved forward another 1/10 meter, so is still in the lead.

Clearly, this continues forever. Achilles reaches a point where the tortoise *used to be*, but the tortoise has already moved ahead some additional distance and so is still in the lead! This happens over and over, infinitely, so Achilles never overtakes the tortoise.

Explain how this paradox can be resolved by using an infinite series. Write down a series that sums up the sequence of distances traveled by Achilles, and use the series to find the exact distance at which Achilles catches up to the tortoise.

2. In class, we showed that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \ln(n+1)$$

We did this using a kind of left Riemann sum for the curve $y = \frac{1}{x}$. Using a right Riemann sum, show that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} < \ln(n)$$

and then deduce that

$$\ln(n+1) < \sum_{i=1}^{n} \frac{1}{i} < \ln(n) + 1$$

Use this fact to estimate $\displaystyle\sum_{i=1}^{1000} \frac{1}{i}$ and $\displaystyle\sum_{i=1}^{1000000} \frac{1}{i}$, as closely as possible. (Obtain both an upper bound and a lower bound for the sum.)

3. I first encountered this sort of problem in *Scientific American Magazine* many years ago…

Sisyphus the inchworm, being punished for his sins, is set down at one end of a one-yard (36 inch), infinitely stretchable rubber band. He crawls along at one inch per minute, and

and he must reach the other end of the rubber band before his punishment ends. But—and here's the nasty bit—at the end of each minute, the rubber band is instantaneously stretched so that it grows by another yard. At the end of the first minute, it stretches to two yards; at the end of the second minute, it stretches to three yards; at the end of the third, it stretches to four yards; and so on forever. Is Sisyphus's punishment everlasting, or will he eventually reach the end?

The bug's only hope is the fact that the rubber band stretches *evenly*, so that the section that is behind Sisyphus stretches proportionately. At the end of the first minute, one inch, or 1/36, of the total distance is behind the inchworm, and 35 inches lie ahead. Then the rubber band stretches to two yards. Now, two inches are behind him—which is still 1/36 of the total distance—and 70 inches lie ahead. So, the *fraction* of the total distance traveled does not change when the rubber band stretches.

a) What fraction of the total distance does Sisyphus cover during the second minute? During the third minute? During the n^{th} minute?

b) Write down a sum that represents the *total* fraction that he has covered after N minutes.

c) Use what you know about a certain infinite series to show that Sisyphus will, in fact, reach the end eventually.

d) Estimate the time that it takes for Sisyphus to compete his punishment.

e) Tantalus is another inchworm, with greater sins. His punishment is similar to Sisyphus's, except that the rubber band at the start is one mile long (1760 yards, or 21,120 inches). Will Tantalus every complete his punishment?

4. For this problem, suppose that a ball is dropped from a height of 10 feet, and that after each bounce it rises to a height that is exactly 1/2 of the height of the previous bounce. So, for the first bounce, the ball rises to a height of 5 feet; for the second, to $\frac{5}{2}$ feet; for the third, to $\frac{5}{4}$ feet, and so on. Let D_i be the total distance traveled—up *and* down—during the i^{th} bounce. We are assuming that the ball bounces infinitely many times. But, does this mean that the ball bounces forever?

a) The total distance traveled by the bouncing ball, from the time it first hits the ground, is $\sum_{i=1}^{\infty} D_i$. Find the distance by finding the sum of the infinite series.

b) When an object is falling freely near the surface of the Earth (ignoring air resistance), its height is given by a function of the form $y = -16t^2 + v_o t + y_o$, where time is measured in seconds, height is measured in feet, v_o is the velocity at time zero, and y_o is the height at time zero. This equations holds for our ball *between bounces.* For a bounce, the initial height is zero, so the formula is $y = -16t^2 + v_o t$. Based on this, and some Calc I, find the *time* taken by each bounce of the ball. To answer this question, you have to solve $y(t) = 0$, and to do that you need to know the value of v_o. (Hint: What is the highest point on the curve $y = -16t^2 + v_o t$?)

c) Write down an infinite series for the total time for which the ball bounces (from the time it first hits the ground). Find the sum of the series, or show that it diverges to infinity. So, does the ball bounce forever?

d) What do you suppose happens with real balls bouncing in the real world?