① $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, so $e^{x^2} = \sum_{k=0}^{\infty} \frac{(x^2)^k}{k!} = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!}$. Integrating gives

$\int e^{x^2} dx = C + \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)k!}$. So, $\int_0^1 e^{x^2} dx = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)k!} \Big|_0^1$

$= \left(\sum_{k=0}^{\infty} \frac{1^{2k+1}}{(2k+1)k!} - 0 \right) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)k!}$.

② $\sin(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}$, so $\cos(x) = \frac{d}{dx}\sin(x) = \frac{d}{dx}\left(\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} \right)$

$= \sum_{m=0}^{\infty} \frac{d}{dx}\left(\frac{(-1)^m x^{2m+1}}{(2m+1)!} \right) = \sum_{m=0}^{\infty} \frac{(-1)^m (2m+1) x^{2m}}{(2m+1)!} = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!}$

③ We know that in a power series $f(x) = \sum_{m=0}^{\infty} c_m (x-a)^m$, it must be true that $c_m = \frac{f^{(m)}(a)}{m!}$. In this case, $c_m = \frac{m}{m^2+1}$ and $a = 3$, so taking $m=10$, $\frac{10}{10^2+1} = c_{10} = \frac{f^{(10)}(3)}{10!}$, and $f^{(10)}(3) = 10! \cdot \frac{10}{10^2+1} = \underline{\underline{\frac{10 \cdot 10!}{101}}}$

④ $\sum_{k=1}^{\infty} \frac{(x-1)^k}{2^k k^m}$: $\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{(x-1)^{k+1} \cdot 2^k k^m}{2^{k+1}(k+1)^m \cdot (x-1)^k} \right|$

$= \lim_{k\to\infty} \left| \frac{x-1}{2} \cdot \left(\frac{k}{k+1}\right)^m \right| = \frac{|x-1|}{2} \cdot \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^m = \frac{|x-1|}{2} \cdot 1 = \frac{|x-1|}{2}$.

So, the series converges for $\frac{|x-1|}{2} < 1$, that is for $|x-1| < 2$, and it diverges for $|x-1| > 2$. This means that the radius of convergence is 2.

⑤ The series in the preceding problem converges for $|x-1| < 2$, i.e. for $-1 < x < 3$, and it diverges for $|x-1| > 2$. The behavior at the endpoints, -1 and 3, depends on the value of m.

a) $\sum_{k=1}^{\infty} \frac{(x-1)^k}{2^k}$ has interval of convergence $(-1, 3)$. [Using $m=0$]

At $x=-1$, the series becomes $\sum_{k=1}^{\infty} \frac{(-2)^k}{2^k} = \sum_{k=1}^{\infty} (-1)^k$.

At $x=3$, it becomes $\sum_{k=1}^{\infty} 1$. Both of these series diverge.

b) $\sum_{k=1}^{\infty} \frac{(x-1)^k}{2^k \cdot k^2}$ has interval of convergence $[-1,3]$. At $x=-1, 3$ the series becomes $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$, $\sum_{k=1}^{\infty} \frac{1}{k^2}$, both of which converge.

c) $\sum_{k=1}^{\infty} \frac{(x-1)^k}{2^k \cdot k}$ has interval of convergence $[-1, 3)$. At $x=-1$, the series is $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$, which converges. For $x=3$, we get $\sum_{k=1}^{\infty} \frac{1}{k}$, which diverges.

d) $\sum_{k=1}^{\infty} \frac{(-1)^k (x-1)^k}{2^k \cdot k}$ has interval of convergence $(-1, 3]$. The $(-1)^k$ reverses the series at the endpoints, compared to part c).

⑥ $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} \frac{1}{a_k}$ cannot both converge. Suppose $\sum_{k=1}^{\infty} a_k$ converges. Then $\lim_{k\to\infty} a_k = 0$, but then $\lim_{k\to\infty} \frac{1}{a_k} = \infty$, so $\sum_{k=1}^{\infty} \frac{1}{a_k}$ diverges by the divergence test. Similarly, if $\sum_{k=1}^{\infty} \frac{1}{a_k}$ converges then $\sum_{k=1}^{\infty} a_k$ diverges. It is possible for both to diverge.
For example, if $a_k = k$ for all k, then $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} k$ and $\sum_{k=1}^{\infty} \frac{1}{a_k} = \sum_{k=1}^{\infty} \frac{1}{k}$. Both of the series diverge in this case.

⑦ a) $\sum_{m=1}^{\infty} \frac{m^2}{m^4+1}$ converges: $\frac{m^2}{m^4+1} < \frac{1}{m^2}$ and $\sum_{m=1}^{\infty} \frac{1}{m^2}$ converges.

b) $\sum_{m=1}^{\infty} \frac{m^2}{m^3+1}$ diverges: $\frac{m^2}{m^3+1} \geq \frac{m^2}{m^3+m^3} = \frac{1}{2m}$, and $\sum_{m=1}^{\infty} \frac{1}{2m}$ diverges.

c) $\sum_{m=1}^{\infty} \frac{m^3}{m^3+1}$ diverges: $\lim_{m\to\infty} \frac{m^3}{m^3+1} = 1 \neq 0$

d) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$ converges by the Alternating Series Test, since $\lim_{k\to\infty} \frac{1}{\sqrt{k}} = 0$ and $\left\{\frac{1}{\sqrt{k}}\right\}_{k=1}^{\infty}$ is decreasing.

(8) $\frac{d}{dx}\int_0^{2x^3} e^{t^2}\,dt = e^{(2x^3)^2}\cdot\frac{d}{dx}2x^3 = 6x^2 e^{4x^6}$

(9) $\Delta x = \frac{1}{2}$. The left Riemann Sum is $f(1)\Delta x + f(1.5)\Delta x + \cdots + f(4.5)\Delta x$

$= \frac{1}{2}[1.3 + 1.6 + 1.85 + 2.12 + 2.4 + 2.6 + 2.77 + 3.0]$.

(A right Riemann Sum would be

$\frac{1}{2}[1.6 + 1.85 + 2.12 + 2.4 + 2.6 + 2.77 + 3.0 + 3.1]$.)

(10) a) $\int \frac{x^2+1}{x^3+3x}\,dx = \frac{1}{3}\int \frac{1}{w}\,dw = \frac{1}{3}\ln|w| + C = \frac{1}{3}\ln|x^3+3x| + C$

Substitution: $w = x^3+3x$, $dw = 3x^2+3\,dx = 3(x^2+1)\,dx$

b) $\int_0^2 2z\sqrt{z^2+1}\,dz = \int_1^5 w^{1/2}\,dw = \frac{2}{3}w^{3/2}\Big|_1^5 = \frac{2}{3}(5^{3/2}-1)$.

Substitution: $w = z^2+1$, $dw = 2z\,dz$, $x=0 \Rightarrow z=1$, $x=2 \Rightarrow z=5$

c) $\int x^2 + xe^{2x}\,dx = \int x^2\,dx + \int xe^{2x}\,dx$

For $\int xe^{2x}\,dx$: by parts

$w = x \qquad dv = e^{2x}\,dx$

$dw = dx \qquad v = \frac{1}{2}e^{2x}$

$= \frac{x^3}{3} + \left[x\cdot\frac{1}{2}e^{2x} - \int\frac{1}{2}e^{2x}\,dx\right]$

$= \frac{x^3}{3} + \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$

d) $\int \sin^2(x)\cos^3(x)\,dx = \int \sin^2(x)\cos^2(x)\cos(x)\,dx = \int \sin^2(x)(1-\sin^2(x))\cos(x)\,dx$

$= \int w^2(1-w^2)\,dw = \int w^2 - w^4\,dw = \frac{w^3}{3} - \frac{w^5}{5} + C = \frac{\sin^3(x)}{3} + \frac{\sin^5(x)}{5} + C$

Substitute: $w = \sin(x)$, $dw = \cos(x)\,dx$

e) $\frac{5x-3}{2x(x-1)} = \frac{A}{2x} + \frac{B}{x-1} \Rightarrow 5x-3 = A(x-1) + B(2x)$

$x=1 \Rightarrow 2 = 2B \Rightarrow B=1$

$x=0 \Rightarrow -3 = -A \Rightarrow A=3$

So $\int \frac{5x-3}{2x(x-1)}\,dx = \int \frac{3}{2x} + \frac{1}{x-1}\,dx = \frac{3}{2}\ln|x| + \ln|x-1| + C$

f) $\int \ln(x^2+1)\,dx$ By parts: $u = \ln(x^2+1)$ $dv = dx$

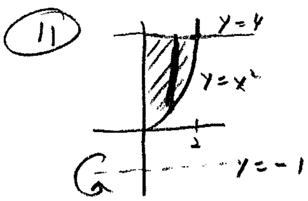
$du = \frac{2x}{x^2+1}dx$ $v = x$

$$= x\ln(x^2+1) - \int x\cdot\frac{2x}{x^2+1}\,dx = x\ln(x^2+1) - 2\int\frac{x^2}{x^2+1}\,dx$$

$$= x\ln(x^2+1) - 2\int\frac{(x^2+1)-1}{x^2+1}\,dx = x\ln(x^2+1) - 2\left[\int 1\,dx - \int\frac{1}{x^2+1}\,dx\right]$$

$$= x\ln(x^2+1) - 2\left(x - \tan^{-1}(x)\right) + C$$

11



Disk method: $\int_a^b \pi R_1^2 - \pi R_2^2\,dx$

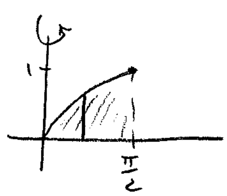
$$\int_0^2 \pi(4+1)^2 - \pi(x^2+1)^2\,dx = \int_0^2 25\pi - \pi(x^4+2x^2+1)\,dx$$

$$= \pi\int_0^2 -x^4-2x^2+24\,dx = \pi\left[-\frac{x^5}{5} - \frac{2x^3}{3} + 24x\Big|_0^2\right]$$

$$= \pi\left[-\frac{32}{5} - \frac{16}{3} + 48\right] = \frac{544}{15}\pi$$

[By shells: $\int_c^d 2\pi r h\,dy = \int_0^4 2\pi(y+1)(\sqrt{y})\,dy$]

12



$$\int_a^b 2\pi r h\,dx = \int_0^{\pi/2} 2\pi x\sin(x)\,dx = 2\pi\int_0^{\pi/2} x\sin(x)\,dx$$

↑ can be done by parts!

$$= 2\pi\left[\sin(x) - x\cos(x)\Big|_0^{\pi/2}\right]$$

$$= 2\pi\left[\left(\sin\left(\tfrac{\pi}{2}\right) - \tfrac{\pi}{2}\cos\tfrac{\pi}{2}\right) - \left(\sin(0) - 0\cos(0)\right)\right]$$

$$= 2\pi\left[\left(1 - \tfrac{\pi}{2}\cdot 0\right) - (0 - 0\cdot 1)\right] = \underline{\underline{2\pi}}$$

13

$$\frac{dy}{dx} = x\sqrt{1-y^2}$$

$$\frac{1}{\sqrt{1-y^2}}\,dy = x\,dx$$

$$\sin^{-1}(y) = \frac{x^2}{2} + C$$

$$y = \sin\left(\frac{x^2}{2} + C\right)$$

plugging in $x=0, y=1$ gives

$$1 = \sin(0+C) = \sin(C)$$

So we can take $C = \frac{\pi}{2}$ and

$$y = \sin\left(\frac{x^2}{2} + \frac{\pi}{2}\right)$$