Math 131-01, Spring 2011 Sample Answers to Lab # 3

(1) a) $\lim_{b\to\infty}\left(\int_1^b \frac{1}{x^2}\,dx\right) = \lim_{b\to\infty}\left(-\frac{1}{x}\Big|_1^b\right) = \lim_{b\to\infty}\left(-\frac{1}{b}-\left(-\frac{1}{1}\right)\right)$

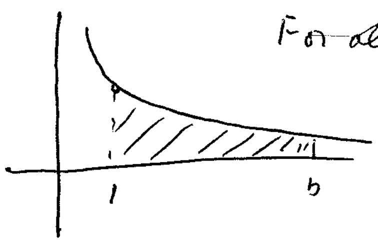
$= \lim_{b\to\infty}\left(1-\frac{1}{b}\right) = 1-0 = \underline{\underline{1}}$

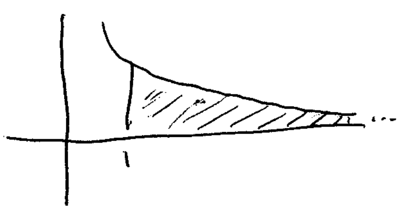
b) $\lim_{b\to\infty}\left(\int_1^b \frac{1}{\sqrt{x}}\,dx\right) = \lim_{b\to\infty}\left(\int_1^b x^{-1/2}\,dx\right) = \lim_{b\to\infty}\left(\frac{x^{1/2}}{1/2}\Big|_1^b\right)$

$= \lim_{b\to\infty}\left(2\sqrt{x}\Big|_1^b\right) = \lim_{b\to\infty}\left(2\sqrt{b}-2\sqrt{1}\right) = \underline{\underline{\infty}}$

c) $\lim_{b\to\infty}\left(\int_1^b \frac{1}{x}\,dx\right) = \lim_{b\to\infty}\left(\ln(x)\Big|_1^b\right) = \lim_{b\to\infty}\left(\ln(b)-\ln(1)\right) = \underline{\underline{\infty}}$

d)





~~For all~~ These problems, the picture looks similar. The integral from 1 to b represents an area as shown. As $b\to\infty$, this area gets bigger, and the limit of the integral seems to represent the area of a region that stretches infinitely to the right. However, for the function $\frac{1}{x^2}$, the total area in the infinitely long region is 1, while for $\frac{1}{x}$ and $\frac{1}{\sqrt{x}}$, it is infinite.

(2) If $\lim_{x\to\infty} f(x) = 1$, then as $b\to\infty$, the average value of $f(x)$ on the interval $[1,b]$ will also approach 1. This is because for all large values of x, $f(x)$ is almost equal to 1. As $b\to\infty$, the "large" values in $[1,b]$, where $f(x)\approx 1$, will occupy a larger and larger fraction of the ~~~~ intervals; x-values where $f(x)$ is <u>not</u> almost 1 will become less and less significant. So, $f(x)$ will be very close to 1 on <u>most</u> of $[1,b]$, and its average value on that interval will also be close to 1.

(3) a) $\int x^{11}\,dx \qquad w = x^6,\quad dw = 6x^5\,dx$

$= \frac{1}{6}\int x^6 \cdot 6x^5\,dx$

$= \frac{1}{6}\int w\,dw$

$= \frac{1}{6}\cdot\frac{1}{2}w^2 + C$

$= \frac{1}{12}\cdot(x^6)^2 + C$

$= \frac{1}{12}x^{12} + C$

b) $\int x^{11}\,dx \qquad w = x^3,\quad dw = 3x^2\,dx$

$= \frac{1}{3}\int x^9 \cdot 3x^2\,dx$

$= \frac{1}{3}\int (x^3)^3 \cdot 3x^2\,dx$

$= \frac{1}{3}\int w^3\,dw$

$= \frac{1}{3}\cdot\frac{1}{4}w^4 + C$

$= \frac{1}{12}\cdot(x^3)^4 + C$

$= \frac{1}{12}x^{12} + C$

c) $\int x^{11}\,dx \qquad w = x^2,\quad dw = 2x\,dx$

$= \frac{1}{2}\int x^{10} \cdot 2x\,dx$

$= \frac{1}{2}\int (x^2)^5 \cdot 2x\,dx$

$= \frac{1}{2}\int w^5\,dw$

$= \frac{1}{2}\cdot\frac{1}{6}w^6 + C$

$= \frac{1}{12}(x^2)^6 + C$

$= \frac{1}{12}x^{12} + C$

(4) $\int \sin(x)\cos(x)\,dx \qquad w = \sin(x),\quad dw = \cos(x)\,dx$

$= \int w\,dw$

$= \frac{1}{2}w^2 + C$

$= \frac{1}{2}\sin^2(x) + C$

$\int \sin(x)\cos(x)\,dx \qquad w = \cos(x),\quad dw = -\sin(x)\,dx$

$= \int -w\,dw$

$= -\frac{1}{2}w^2 + C$

$= -\frac{1}{2}\cos^2(x) + C$

Antiderivatives are not unique, but any two antiderivatives of the same function must differ by a constant. (That's the meaning of the "$+C$" in the answer.). In this case,

$$\left(\frac{1}{2}\sin^2(x)\right) - \left(-\frac{1}{2}\cos^2(x)\right) = \frac{1}{2}\left(\sin^2(x) + \cos^2(x)\right)$$

$$= \frac{1}{2}\cdot 1 = \frac{1}{2}$$

So the two antiderivatives that we have found <u>do</u> differ by a constant. Another way to look at this is to use the fact that $1 = \sin^2(x) + \cos^2(x)$ to write

$$\frac{1}{2}\sin^2(x) - \frac{1}{2} = \frac{1}{2}\sin^2(x) - \frac{1}{2}\left(\sin^2(x) + \cos^2(x)\right)$$

$$= \frac{1}{2}\sin^2(x) - \frac{1}{2}\sin^2(x) - \frac{1}{2}\cos^2(x) = -\frac{1}{2}\cos^2(x)$$

so we can get one antiderivative by adding $-\frac{1}{2}$ to the other.

(4) a) $\int \frac{x^3}{\sqrt{2x^2+1}}\,dx$ $w = 2x^2+1$, $dw = 4x\,dx$ $x^2 = \frac{1}{2}(w-1)$

$$= \frac{1}{4}\int \frac{1}{\sqrt{2x^2+1}} \cdot x^2 \cdot 4x\,dx = \frac{1}{4}\int \frac{1}{\sqrt{w}} \cdot \frac{1}{2}(w-1)\,dw$$

$$= \frac{1}{8}\int w^{1/2} - w^{-1/2}\,dw = \frac{1}{8}\left(\frac{w^{3/2}}{3/2} - \frac{w^{1/2}}{1/2}\right) + C$$

$$= \frac{1}{8}\left(\frac{2}{3}(2x^2+1)^{3/2} - 2(2x^2+1)^{1/2}\right) + C$$

b) $\int \frac{1}{1-\sqrt{x}}\,dx$ $w = 1-\sqrt{x}$, $dw = -\frac{1}{2\sqrt{x}}\,dx$ $\Rightarrow \sqrt{x} = 1-w$ $\Rightarrow dx = -2\sqrt{x}\,dw = -2(1-w)\,dw$

$$= \int \frac{1}{w}\cdot(-2(1-w))\,dw$$

$$= -2\int \frac{1}{w} - 1\,dw = -2\left(\ln|w| - w\right) + C$$

$$= \cancel{-2\left(\ln|1-\sqrt{x}| - (1-\sqrt{x})\right) + C}$$

$$= -2\left(\ln|1-\sqrt{x}| - (1-\sqrt{x})\right) + C$$

c) $\int \frac{1}{\sqrt{x}(1+x)}\,dx$ $w = \sqrt{x}$, $dw = \frac{1}{2\sqrt{x}}\,dx$, $2\,dw = \frac{1}{\sqrt{x}}\,dx$ $\Rightarrow w^2 = x$

$$= \int \frac{1}{1+x}\cdot\frac{1}{\sqrt{x}}\,dx$$

$$= \int \frac{1}{1+w^2}\cdot 2w = 2\tan^{-1}(w) + C = 2\tan^{-1}(\sqrt{x}) + C$$

d) $\int \sin^3(x)\,dx$ $w = \cos(x)$, $dw = -\sin(x)\,dx$ $\sin^2(x) = 1-\cos^2(x)$

$$= \int \sin^2(x)\cdot\sin(x)\,dx$$

$$= -\int (1-\cos^2(x))\cdot(-\sin(x))\,dx = -\int(1-w^2)\,dw = -\left(w - \frac{1}{3}w^3\right) + C$$

$$= -\left(\cos(x) - \frac{1}{3}\cos^3(x)\right) + C$$