(1) a) Can be answered with either a left or a right Riemann Sum.

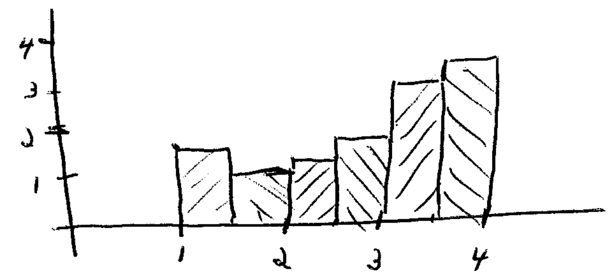
Here is a left Riemann sum: $\Delta x = \frac{1}{2}$; $x_0 = 1, x_1 = 1.5, x_2 = 2, \ldots, x_m = 4$

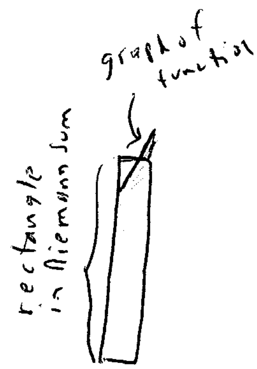
$$\sum_{k=1}^{m} f(x_{k-1})\,\Delta x = \sum_{k=1}^{6} f(x_{k-1}) \cdot \frac{1}{2}$$

$$= \Big(f(1)\cdot\tfrac{1}{2} + f(1.5)\cdot\tfrac{1}{2} + f(2)\cdot\tfrac{1}{2} + f(2.5)\cdot\tfrac{1}{2} + f(3)\cdot\tfrac{1}{2} + f(3.5)\cdot\tfrac{1}{2}\Big)$$

$$= 1.5\cdot\tfrac{1}{2} + 1\cdot\tfrac{1}{2} + 1.2\cdot\tfrac{1}{2} + 1.7\cdot\tfrac{1}{2} + 2.9\cdot\tfrac{1}{2} + 3.3\cdot\tfrac{1}{2}$$

b)





c) As the rectangles become narrower, the difference between the actual area under the curve and the areas of the rectangles becomes smaller and smaller. So as $n \to \infty$, the error in the approximation goes to zero.

(2) This is problem #3!!

a) $$\int_1^4 4x^3 - 6x\,dx = x^4 - 3x^2 \Big|_1^4 = (4^4 - 3\cdot 4^2) - (1^4 - 3\cdot 1^2)$$

$$= 256 - 48 - 1 + 3$$

$$= \underline{\underline{210}}$$

b) $$\int_0^2 x\sqrt{4-x^2}\,dx$$

$w = 4 - x^2$

$dw = -2x\,dx$

$-\frac{1}{2}dw = x\,dx$

$x = 0 \Rightarrow w = 4$

$x = 2 \Rightarrow w = 0$

$$= -\frac{1}{2}\int_4^0 w^{1/2}\,dw$$

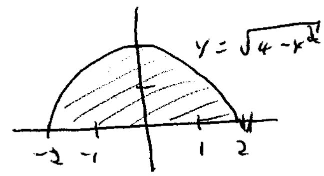
$$= -\frac{1}{2}\cdot\frac{w^{3/2}}{3/2}\Big|_4^0$$

$$= -\frac{1}{3}w^{3/2}\Big|_4^0$$

$$= \left(-\frac{1}{3}\cdot 0\right) - \left(-\frac{1}{3}\cdot 4^{3/2}\right)$$

$$= \frac{1}{3}\cdot 8 = \underline{\underline{\frac{8}{3}}}$$

c) The function $f(x)=\sqrt{4-x^2}$, for $-2 \le x \le 2$, is half of a circle of radius 2. ($y=\sqrt{4-x^2} \Rightarrow y^2 = 4-x^2$
$\Rightarrow x^2+y^2=4$, which we recognize as the equation of a circle.) So,



The integral $\int_{-2}^{2} \sqrt{4-x^2}\,dx$ represents the area of the semicircle, which is $\frac{1}{2}\cdot\pi r^2 = \frac{1}{2}\cdot\pi\cdot 2^2 = \underline{\underline{2\pi}}$.

② This is problem #2

a) $\int \cos(6x)\,dx$ $\quad w = 6x$, $dw = 6\,dx$

$= \frac{1}{6}\int \cos(6x)\cdot 6\,dx$

$= \frac{1}{6}\int \cos(w)\,dw$

$= \frac{1}{6}\sin(w) + C$

$= \underline{\underline{\frac{1}{6}\sin(6x) + C}}$

b) $\int 4t^5(1+\sqrt{t})\,dt$

$= \int 4t^5 + 4t^{5+1/2}\,dt$

$= \int 4t^5\,dt + \int 4t^{11/2}\,dt$

$= 4\cdot\frac{1}{6}t^6 + 4\cdot\frac{2}{13}t^{13/2} + C$

$= \underline{\underline{\frac{2}{3}t^6 + \frac{8}{13}t^{13/2} + C}}$

c) $\int e^x(4+e^x)\,dx$

$= \int 4e^x + e^{2x}\,dx$

$= \underline{\underline{4e^x + \frac{1}{2}e^{2x} + C}}$

$\underline{\underline{\text{OR}}}$ $\int e^x(4+e^x)\,dx$ $\quad w = 4+e^x$, $dw = e^x\,dx$

$= \int (4+e^x)\cdot e^x\,dx$

$= \int w\,dw$

$= \frac{1}{2}w^2 + C$

$= \underline{\underline{\frac{1}{2}(4+e^x)^2 + C}}$

d) $\int \frac{(\ln(x))^3}{x}\,dx$ $\quad w = \ln x$, $dw = \frac{1}{x}\,dx$

$= \int w^3\cdot dw$

$= \frac{1}{4}w^4 + C$

$= \underline{\underline{\frac{1}{4}(\ln(x))^4 + C}}$

e) $\int \frac{\sec^2(x)}{1+\tan(x)}\,dx$ $\quad w = 1+\tan(x)$, $dw = \sec^2(x)\,dx$

$= \int \frac{1}{w}\,dw$

$= \ln(|w|) + C$

$= \underline{\underline{\ln(|1+\tan(x)|) + C}}$

f) $\int (x+1)\sqrt{x^2+2x+3}\,dx$

$u = x^2+2x+3$

$du = (2x+2)\,dx$

$= 2(x+1)\,dx$

$\frac{1}{2}du = (x+1)\,dx$

$= \int \sqrt{x^2+2x+3}\cdot(x+1)\,dx$

$= \int u^{1/2}\cdot\frac{1}{2}du$

$= \frac{1}{2}\cdot\frac{u^{3/2}}{3/2}+C = \frac{1}{2}\cdot\frac{2}{3}\cdot u^{3/2}+C = \underline{\frac{1}{3}(x^2+2x+3)^{3/2}+C}$

4) a) We know that $\int_0^5 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^5 f(x)\,dx$,

so $\int_3^5 f(x)\,dx = \int_0^5 f(x)\,dx - \int_0^3 f(x)\,dx = 7-10 = \underline{-3}$

b) Since $\int_0^3 f(x)\,dx$ is negative, it's impossible for $f(x)$ to be greater than 0 for all x. A definite integral of a function that lies above the x-axis represents an area, and so must be positive.

5) $\frac{1}{5-1}\int_1^5 \sqrt{x}\,dx = \frac{1}{4}\cdot\frac{2}{3}x^{3/2}\Big|_1^5 = \frac{1}{6}(5^{3/2}-1^{3/2}) = \underline{\frac{1}{6}(5^{3/2}-1)}$

6) $\frac{d}{dx}\int_0^{3x}\sqrt{z^3+1}\,dz = \underbrace{\sqrt{(3x)^3+1}}_{\text{by the F.T.o.C.}}\cdot\underbrace{\frac{d}{dx}(3x)}_{\text{by the chain rule}} = \underline{\sqrt{(3x)^3+1}\cdot 3}$

7) $|f(x)| \geq 0$ for all x, so $\int_a^b |f(x)|\,dx$ represents an area, which is always non-negative. Alternatively, you could note that all the terms in a Riemann Sum for $|f(x)|$ are ≥ 0, so the limit of the Riemann Sums must also be ≥ 0.

8) $\sum_{i=1}^{5}\frac{3i}{i^2+1} = \frac{3\cdot1}{1^2+1}+\frac{3\cdot2}{2^2+1}+\frac{3\cdot3}{3^2+1}+\frac{3\cdot4}{4^2+1}+\frac{3\cdot5}{5^2+1}$

(9) $\Delta x = \frac{2-0}{6} = \frac{1}{3}$; $x_0 = 0,\ x_1 = \frac{1}{3},\ x_2 = \frac{2}{3},\ x_3 = \frac{3}{3} = 1,\ x_4 = \frac{4}{3},\ x_5 = \frac{5}{3},\ x_6 = \frac{6}{3} = 2.$

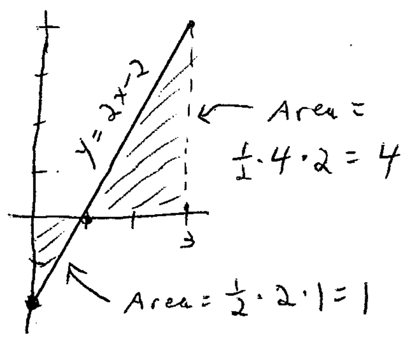
$$\sum_{k=1}^{6} f(x_k)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x$$

$$= f(\tfrac{1}{3})\cdot\tfrac{1}{3} + f(\tfrac{2}{3})\cdot\tfrac{1}{3} + f(1)\cdot\tfrac{1}{3} + f(\tfrac{4}{3})\cdot 1 + f(\tfrac{5}{3})\cdot 1 + f(2)\cdot 1$$

$$= \sqrt{(\tfrac{1}{3})^3+1}\cdot\tfrac{1}{3} + \sqrt{(\tfrac{2}{3})^3+1}\cdot\tfrac{1}{3} + \sqrt{(1)^3+1}\cdot\tfrac{1}{3}$$

$$+ \sqrt{(\tfrac{4}{3})^3+1}\cdot\tfrac{1}{3} + \sqrt{(\tfrac{5}{3})^3+1}\cdot\tfrac{1}{3} + \sqrt{(2)^3+1}\cdot\tfrac{1}{3}$$

(10) a)



$$\int_0^3 2x-2\,dx = 4 - 1 = \underline{\underline{3}}.$$

The integral can be computed as the area above the axis minus the area below the axis.

b) $\int_0^3 2x-2\,dx = \lim_{n\to\infty}\left(\sum_{k=1}^{n} f(x_k)\Delta x\right)$

For a given n, $\Delta x = \frac{3-0}{n} = \frac{3}{n}$ and $x_k = a + k\cdot\Delta x = 0 + k\cdot\frac{3}{n} = \frac{3k}{n}$.

So $\sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n} f\left(\frac{3k}{n}\right)\cdot\frac{3}{n} = \sum_{k=1}^{n}\left(2\cdot\frac{3k}{n} - 2\right)\cdot\frac{3}{n}$

$$= \sum_{k=1}^{n}\left(\frac{18k}{n^2} - \frac{6}{n}\right) = \left(\sum_{k=1}^{n}\frac{18k}{n^2}\right) - \left(\sum_{k=1}^{n}\frac{6}{n}\right)$$

$$= \left(\frac{18}{n^2}\cdot\sum_{k=1}^{n} k\right) - \left(\frac{6}{n}\cdot\sum_{k=1}^{n} 1\right)$$

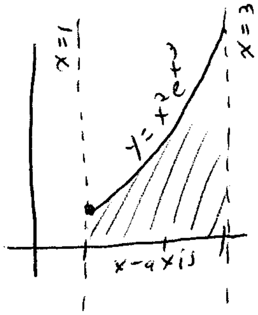
$$= \left(\frac{18}{n^2}\cdot\frac{n(n+1)}{2}\right) - \left(\frac{6}{n}\cdot n\right)$$

$$= \frac{9(n+1)}{n} - 6$$

As $n \to \infty$, this approaches $9-6$ or $\underline{\underline{3}}$. So,

$$\int_0^3 2x-2\,dx = \lim_{n\to\infty}\left(\sum_{k=1}^{n} f(x_k)\Delta x\right) = \lim_{n\to\infty}\left(\frac{9(n+1)}{n} - 6\right) = \underline{\underline{3}}.$$

(11)



This is just another way of asking for the value of the definite integral $\int_1^3 x^2 e^{x^3} dx$

$$\int_1^3 x^2 e^{x^3} dx$$

$w = x^3$
$dw = 3x^2 dx$
$\frac{1}{3} dw = x^2 dx$
when $x=1$, $w=1$
when $x=3$, $w=27$

$$= \int_1^{27} e^w \cdot \frac{1}{3} dw$$

$$= \frac{1}{3} e^w \Big|_1^{27}$$

$$= \frac{1}{3} e^{27} - \frac{1}{3} e^1 = \underline{\frac{1}{3}(e^{27} - e)}$$

(12) The area under a curve is a geometric concept. For non-trivial curves, we can approximate the area using rectangular strips (giving a Riemann Sum), and take the limit of this approximation as the width of the rectangles approaches zero. This gives the definition of the definite integral $\int_a^b f(x)\,dx$. Nothing in this definition uses the concept of derivative or antiderivative.

However, according to the Fundamental Theorem of Calculus, the definite integral is closely related to antiderivatives. First of all, if we know an antiderivative $F(x)$ for a continuous function $f(x)$ [That is $F'(x) = f(x)$], we can use it to find the value of a definite integral of f:

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Second, if we <u>don't</u> have an antiderivative for f, we can define one using the definite integral:

If $A(x) = \int_a^x f(t)\,dt$, then $A'(x) = f(x)$.