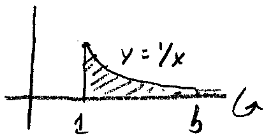
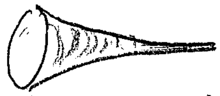
① (At the end)

② a)



$$V = \int_1^b \pi \cdot \left(\tfrac{1}{x}\right)^2 dx = \pi \int_1^b x^{-2}\, dx = \pi \cdot \left(-x^{-1}\right)\Big|_1^b$$
$$= \pi\left(-\tfrac{1}{b} - (-1)\right) = \underline{\pi\left(1 - \tfrac{1}{b}\right)}$$



The solid looks like the bell of a trumpet

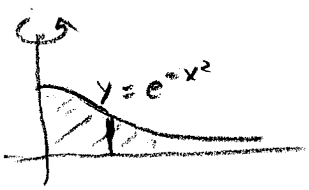
(b) $$\lim_{b\to\infty} \int_1^b \pi \cdot \left(\tfrac{1}{x}\right)^2 dx = \lim_{b\to\infty} \pi\left(1 - \tfrac{1}{b}\right)$$
$$= \pi(1-0) = \underline{\underline{\pi}}$$

So the infinite horn has finite volume π

c) I believe that there only seems to be a paradox because of confusion about paint!

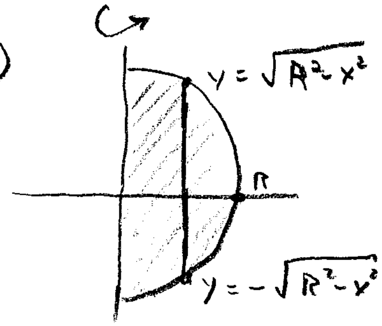
- Real physical paint cannot be used to fill the trumpet because the diameter of the trumpet approaches 0 as $x \to \infty$. At some point, it becomes too narrow for even a single molecule of paint. The paint that you can fit in, up to that point, only covers a finite part of the infinite area under the curve (an infinite area remains unpainted). So there is no paradox.
- If you imagine an ideal paint that can fill the arbitrily thin tube, then a finite amount of that paint can cover an infinite area. (Use $\frac{1}{2}$ of the paint on the 1^{st} square foot, $\frac{1}{4}$ on the 2^{nd} square foot, $\frac{1}{8}$ on the 3^{rd}, etc.) Again, there is no paradox.

③



$$\lim_{b\to\infty}\left(\int_0^b 2\pi x e^{-x^2} dx\right) = \lim_{b\to\infty}\left[-\pi e^{-x^2}\Big|_0^b\right]$$
$$= \lim_{b\to\infty}\left(-\pi \cdot \left(e^{-b^2} - e^0\right)\right)$$
$$= \lim_{b\to\infty}\left(-\pi\left(e^{-b^2} - 1\right)\right)$$
$$= \lim_{b\to\infty} \pi\left(1 - \frac{1}{e^{b^2}}\right)$$
$$= \underline{\underline{\pi}}$$

(4) a)



rotate about x-axis

$$\int_0^R 2\pi \cdot x \cdot \left(2\sqrt{R^2-x^2}\right) dx$$

$$= 2\pi \int_0^R 2x\sqrt{R^2-x^2}\, dx$$

$w = R^2 - x^2 \qquad x = 0 \Rightarrow w = R^2$

$dw = -2x\,dx \qquad x = R \Rightarrow w = 0$

$$= 2\pi \int_{R^2}^{0} -w^{1/2}\, dw$$

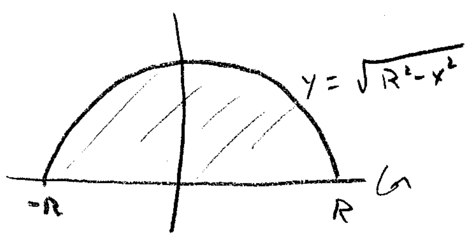
$$= 2\pi \left(-\frac{2}{3} w^{3/2} \Big|_{R^2}^{0}\right)$$

$$= 2\pi \left(0 - \left(-\frac{2}{3}(R^2)^{2/3}\right)\right)$$

$$= 2\pi \left(\frac{2}{3} R^3\right)$$

$$= \frac{4}{3}\pi R^3$$

b)



rotate about the x-axis

$$\int_{-R}^{R} \pi \left(\sqrt{R^2-x^2}\right)^2 dx$$

$$= \pi \int_{-R}^{R} (R^2 - x^2)\, dx$$

$$= \pi \left(R^2 x - \frac{1}{3}x^3 \Big|_{-R}^{R} \right)$$

$$= \pi \left(\left(R^2 \cdot R - \frac{1}{3}R^3\right) - \left(-R^2 \cdot R - \frac{1}{3}R^3\right) \right)$$

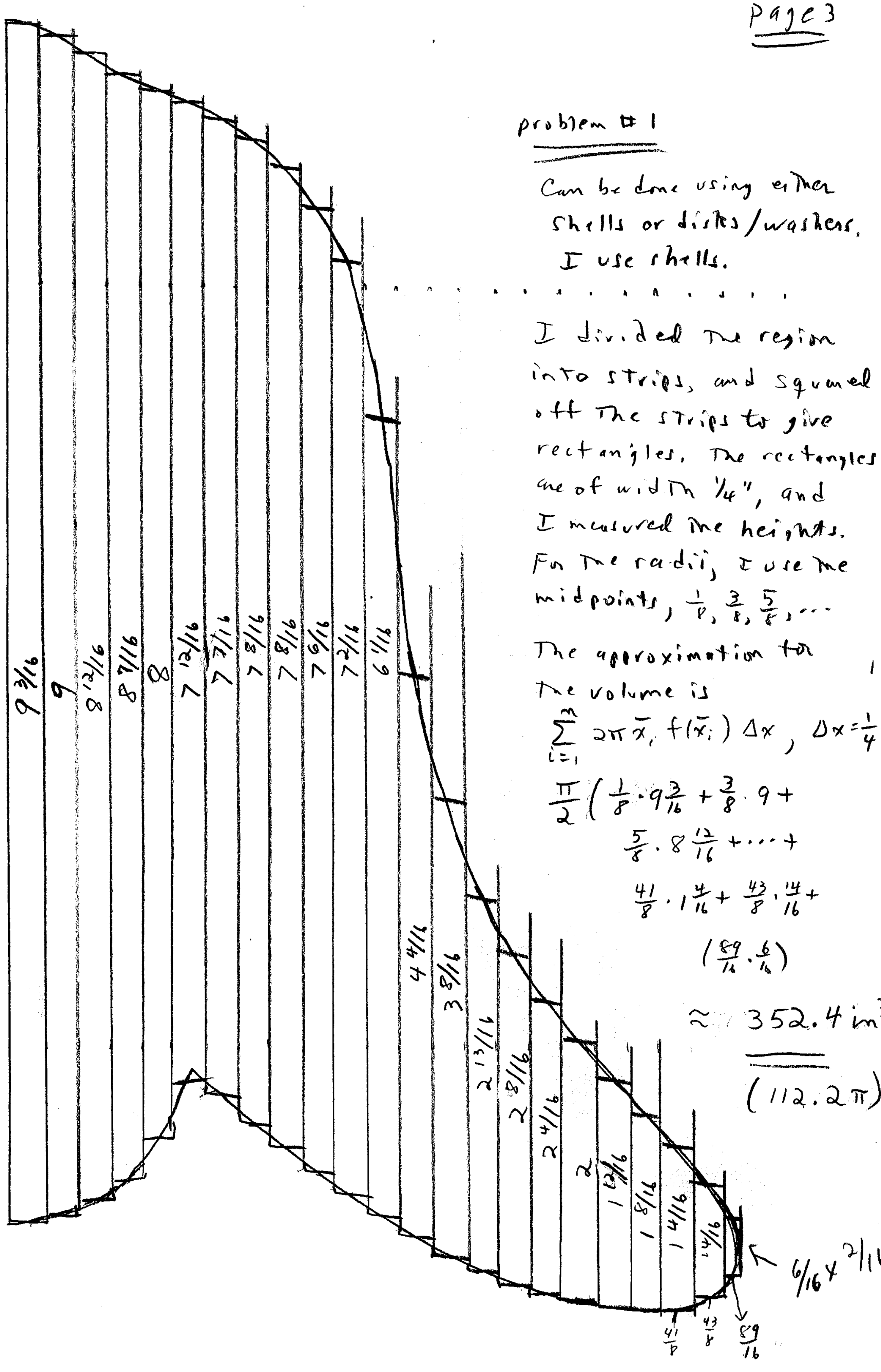
$$= \pi \left(\frac{2}{3}R^3 - \left(-\frac{2}{3}R^3\right) \right)$$

$$= \frac{4}{3}\pi R^3$$

c)

$$\int_0^R 4\pi r^2\, dr = 4\pi \cdot \frac{1}{3} r^3 \Big|_0^R = \frac{4}{3}\pi R^3$$

[This makes the sphere from concentric spherical surfaces of radius r, with r ranging from 0 up to R, similar to a disk made of concentric circles.]



problem # 1

Can be done using either shells or disks/washers. I use shells.

. . . .

I divided the region into strips, and squared off the strips to give rectangles. The rectangles are of width 1/4", and I measured the heights. For the radii, I use the midpoints, $\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \ldots$

The approximation for the volume is

$$\sum_{i=1}^{n} 2\pi \bar{x}_i f(\bar{x}_i)\,\Delta x, \quad \Delta x = \frac{1}{4}$$

$$\frac{\pi}{2}\Big(\frac{1}{8}\cdot 9\tfrac{3}{16} + \frac{3}{8}\cdot 9 + \frac{5}{8}\cdot 8\tfrac{12}{16} + \cdots + \frac{41}{8}\cdot 1\tfrac{4}{16} + \frac{43}{8}\cdot\frac{14}{16} + \Big(\frac{89}{16}\cdot\frac{6}{16}\Big)$$

$$\approx 352.4\ \text{in}^3$$

(112.2π)