① a) $\frac{dy}{dt} = \frac{t^2}{y}$

$y\,dy = t^2\,dt$

$\int y\,dy = \int t^2\,dt$

$\frac{1}{2}y^2 = \frac{1}{3}t^3 + C_1$

$y^2 = \frac{2}{3}t^3 + C$ (where $C = 2C_1$)

$\underline{y = \pm\sqrt{\frac{2}{3}t^3 + C}}$

b) $2ty\frac{dy}{dt} = 1$

$2y\,dy = \frac{1}{t}\,dt$

$\int 2y\,dy = \int \frac{1}{t}\,dt$

$y^2 = \ln|t| + C$

$\underline{y = \pm\sqrt{\ln|t| + C}}$

c) $\frac{dy}{dt} = \cos^2(y)\cos(t)$

$\frac{1}{\cos^2(y)}\,dy = \cos(t)\,dt$

$\sec^2(y)\,dy = \cos(t)\,dt$

$\int \sec^2(y)\,dy = \int \cos(t)\,dt$

$\tan(y) = \sin(t) + C$

$\underline{y = \tan^{-1}(\sin(t) + C)}$

② a) $\frac{dP}{dt} = -1.2P$, $P(0) = 7$. $\frac{dP}{dt} = -1.2P$ is an example of exponential decay, with solution $P(t) = Ae^{-1.2t}$, where $A = P(0) = 7$. So the solution is $\underline{P(t) = 7e^{-1.2t}}$

b) $2ty\frac{dy}{dt} = 1$, $y(1) = 3$. From problem 1a, the general solution is $y = \pm\sqrt{\ln|t| + C}$. Since $y(1) = 3$, we must have $3 = \pm\sqrt{\ln(1) + C} = \pm\sqrt{C}$.

Squaring gives $C = 9$. So the solution is

$y = \sqrt{\ln|t| + 9}$. (use positive square root since $y(1) > 0$.)

③ $y(t) = Ae^{kt}$. $y(t + \frac{\ln(2)}{k}) = Ae^{k(t + \frac{\ln(2)}{k})} = Ae^{kt + \ln(2)}$

$= Ae^{kt} \cdot e^{\ln(2)} = Ae^{kt} \cdot 2 = y(t) \cdot 2$. So if you start at any time t and check again $\frac{\ln(2)}{k}$ time units later, the amount or size of y has doubled.

④ The amount after t years is $P(t) = Ae^{rt}$, where $A = \$1000$ (the initial amount) and $r = 0.08$, so $P(t) = 1000e^{0.08t}$ and

$P(1) = 1000\,e^{0.08t} = 1000 \times 1.083287 = \underline{\$1083.29}$. This is more than $1080 because the interest is compounded — after some interest has been earned, the interest starts earning interest.

(doubling time $= \frac{\ln(2)}{r} = \frac{\ln(2)}{0.08} = 8.664$ years)

⑤ We know that if $y(t)$ is the value t years after 1907, then $y(t) = 25e^{rt}$, where r is the interest rate.

Since 2011 is 104 years after 1907 and $y(t) = 1300.98$ in 2011, we have $y(104) = 1300.98$. So $1300.98 = 25e^{r \cdot 104}$, we can solve this for r: $\frac{1300.98}{25} = e^{104r}$,

$\ln\left(\frac{1300.98}{25}\right) = \ln(e^{104r}) = 104r$, so $r = \frac{1}{104}\ln\left(\frac{1300.98}{25}\right)$.

This is about $r = 0.03799997$. As a percentage, it looks like r was exactly $r = \underline{3.8\%}$

⑥ The half-life is $\frac{\ln(2)}{k}$! $y(t + \frac{\ln(2)}{k}) = Ae^{-k(t + \frac{\ln(2)}{k})}$

$= Ae^{-kt - \ln(2)} = Ae^{-kt}e^{-\ln(2)} = Ae^{-kt} \cdot \frac{1}{e^{\ln(2)}} = Ae^{-kt} \cdot \frac{1}{2}$

$= y(t) \cdot \frac{1}{2}$. So, after $\frac{\ln(2)}{k}$ time units, the amount has been cut to $\frac{1}{2}$ of the original amount.

⑦ If $y(t)$ is the amount left after t years, we know $y(0) = 1000$ g, so $y(t) = 1000\,e^{-kt}$, where k is the decay rate. We also know $y(1) = 1000 - 0.037 = 999.963$, so $999.963 = 1000\,e^{-k \cdot 1}$, or $0.999963 = e^{-k}$ and $-k = \ln(0.99963) = -0.00003700068$, and $k = 0.00003700068$. Half life $= \frac{\ln(2)}{k} = \underline{18{,}733.361 \text{ years}}$.