This lab is a little different from previous ones, because it is a "learning lab." It introduces some new ideas that have not been covered in class and asks you to work with them. (The ideas, however, are simple extensions of things that we have done or that were done in Calculus I.)

Because of Spring Break, the lab will be due two weeks from today. All problems should be turned in.

Recall that a *differential equation* is an equation involving an unknown function and its derivatives. In a "first-order" equation, only the first derivative appears. A *solution* of a differential equation is a function that makes the equation true. In general, there is an infinite family of possible solutions. An important example of a differential equation is

$$\frac{dy}{dt} = ky$$

where k is a constant. For $k > 0$, this is the equation for *exponential growth*, and for $k < 0$, it represents *exponential decay*. This equation can be solved by "separation of variables," that is, moving all occurrences of one variable to one side of the equation and all occurrences of the other variable to the other side:

$$\frac{1}{y}\,dy = k\,dt$$

Once the variables have been separated, integrate both sides to find the solution:

$$\int \frac{1}{y}\,dy = \int k\,dt$$

In this case, we get $\ln|y| = kt + C$, where C is a constant. This can be solved for y to give the function $y = \pm e^{kt+C} = Ae^{kt}$, where A represents the constant $\pm e^{C}$. That is, any solution to the exponential growth and decay equation is of the form $y = Ae^{kt}$ for some constant A. (The equation for exponential decay is often written $\frac{dy}{dt} = -kt$, where $k > 0$, and the solution as $y = Ae^{-kt}$.)

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1. The method of separation of variables can be applied to other differential equations in addition to the exponential growth and decay equation. Use separation of variables to find all solutions to the following differential equations. Your answer should be in the form $y = f(t)$, where the function $f(t)$ will involve some arbitrary constant.

a) $\dfrac{dy}{dt} = \dfrac{t^2}{y}$ **b)** $2ty\dfrac{dy}{dt} = 1$ **c)** $\dfrac{dy}{dt} = \cos^2(y)\cos(t)$

Often, we consider "initial value problems," where the value of the solution at a point is given. The initial value picks out one particular solution from the infinite family of solutions.

For example, we might want to solve $\frac{dy}{dt} = 2.37\,y$, with an initial condition $y(0) = 200$. We know the general solution $y(t) = Ae^{2.37t}$, for some constant A. Using the initial condition, we get that $200 = y(0) = Ae^{2.37\cdot 0} = A \cdot 1 = A$, so in this case A is just the initial value at time $t = 0$, and the solution of the initial value problem is $y = 200e^{2.37t}$.

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2. Solve the initial value problems

a) $\dfrac{dP}{dt} = -1.2P$, with $P(0) = 7$ **b)** $2ty\dfrac{dy}{dt} = 1$, with $y(1) = 3$

Exponential growth and decay are common in real applications. For example, the decay of a radioactive isotope is modeled very precisely by an exponential decay equation. Population tends to grow exponentially, at least in the short term. And money earning interest in a bank account grows exponentially. There are many interesting problems based on exponential growth and decay, such as the remaining problems on this lab...

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3. Suppose that a quantity $y(t)$ grows exponentially according to the equation $y(t) = Ae^{kt}$, where $k > 0$. Show that the value of y doubles in any time period of length $\frac{\ln(2)}{k}$. That is show that for any t, $y\big(t + \frac{\ln(2)}{k}\big) = 2 \cdot y(t)$. The number $\frac{\ln(2)}{k}$ is referred to as the *doubling time* for $y(t)$.

4. When a bank account earns interest "compounded continuously" at an interest rate of r, it means that the amount, $P(t)$, in the account grows exponentially according to the equation $P(t) = Ae^{rt}$. (Time measured in years, and r representing the annual interest rate given as a decimal, not a percentage.) If \$1000 is invested at an 8% interest rate, compounded continuously, how much is it worth at the end of one year? Explain why the answer is not \$1080. What is the doubling time for $P(t)$?

5. A man finds an old savings bond that was purchased in 1907 for \$25. Since then, it has been earning interest at an annual interest rate r, compounded continuously. Assuming that the bond is now worth \$1300.98, find r.

6. Exponential decay is exhibited by the differential equation $\frac{dy}{dx} = -kt$, where $k > 0$, and its solutions $y(t) = Ae^{-kt}$. Instead of a doubling time, a quantity that decays exponentially exhibits a *half-life*, that is, a time period in which the quantity decreases by half. Prove this, and find a formula for the half-life.

7. Radioactive decay is an example of exponential decay. Over the course of a year, 0.037 grams decays, out of an initial 1 kilogram sample of a certain radioactive isotope. What is the half-life of the isotope?