(1) a) The frictional force acts in the opposite direction to the velocity; that is, it tries to slow a moving object down. In this case, the object is moving downwards, so the friction force points up.

b)
$$y(t) = \frac{32}{k}t + \frac{32}{k^2}(e^{-kt} - 1)$$

$$y'(t) = \frac{32}{k} + \frac{32}{k^2}\cdot\left(e^{-kt}\cdot(-k)\right)$$

$$= \frac{32}{k} - \frac{32}{k}e^{-kt}$$

$$y''(t) = -\frac{32}{k}\cdot e^{-kt}\cdot(-k)$$

$$= 32e^{-kt}$$

Note: As $t \to \infty$, $y'(t) \to \frac{32}{k}$ and $y''(t) \to 0$

So, $32 - ky' = 32 - k\cdot\left(\frac{32}{k} - \frac{32}{k}e^{-kt}\right)$

$$= 32 - 32 + 32e^{-kt}$$

$$= 32e^{-kt}$$

$$= y'' \checkmark$$

So $y(t)$ does satisfy the equation $y'' = 32 - ky'$

c) As t goes to infinity, the acceleration approaches 0 and the velocity approaches the constant $\frac{32}{k}$. As the velocity increases, so does the force of friction. When the friction force just balances the force of gravity, the net force is zero, the acceleration is 0, and the velocity stops changing. Mathematically, the acceleration doesn't actually become equal to zero, but approaches it as a limit.

As $t \to \infty$, the distance fallen approaches the linear function $y = \frac{32}{k}t + \frac{32}{k^2}$ asymptotically,

d) The terminal velocity is $\frac{32}{k}$. The difference between $y(t)$ and the terminal velocity is $\left| y(t) - \frac{32}{k} \right|$

$$= \left| \left(\frac{32}{k} - \frac{32}{k}e^{-kt}\right) - \frac{32}{k} \right| = \frac{32}{k} e^{-kt}.$$ This is of the form Ae^{-kt}, which is exponential decay,

e) The terminal velocity, $\frac{32}{k}$, depends on k, which in turn depends on the velocity of the falling object. For an ant, k is much bigger than for a person, and its terminal velocity is smaller. So the person hits the ground at a much higher velocity than the ant. (If you squashed the person to be the same thickness as the ant, they would have the same terminal velocities — but I guess the person wouldn't come out of it any better.)

2 a) $\int \sin(x)\cos(x)\, e^{\sin(x)}\, dx$ $\qquad z = \sin(x)$, $dz = \cos(x)\, dx$

$$= \int z e^z\, dz \qquad u = z,\ du = dz \qquad dv = e^z dz,\ v = e^z$$

$$= ze^z - \int e^z\, dz$$

$$= ze^z - e^z + C = \underline{\sin(x)\, e^{\sin(x)} - e^{\sin(x)} + C}$$

b) $\int \frac{\ln(\ln(x))}{x}\, dx$ $\qquad z = \ln(x)$, $dz = \frac{1}{x} dx$

$$= \int \ln(z)\, dz \qquad u = \ln(z),\ du = \frac{1}{z} dz \qquad dv = dz,\ v = z$$

$$= z\ln(z) - \int z \cdot \frac{1}{z} \cdot dz = z \ln z - \int 1\, dz$$

$$= z \ln z - z + C = \underline{\ln(x)\ln(\ln(x)) - \ln(x) + C}$$

c) $\int \sin(\sqrt{x})\,dx$ $\qquad \theta = \sqrt{x}$, $\quad d\theta = \frac{1}{2\sqrt{x}}\,dx$

$= \int \sin(\theta)\cdot 2\theta\,d\theta \qquad dx = 2\sqrt{x}\,d\theta = 2\theta\,d\theta$

$= 2\int \theta \sin\theta\,d\theta \qquad u = \theta \quad dv = \sin\theta\,d\theta$

$du = d\theta \qquad v = -\cos\theta\,d\theta$

$= 2\left[\theta\cdot(-\cos\theta) - \int(-\cos\theta)\cdot d\theta\right]$

$= -2\theta\cos\theta + 2\int \cos(\theta)\,d\theta$

$= -2\theta\cos\theta + 2\sin\theta + C$

$= \underline{-2\sqrt{x}\,\cos(\sqrt{x}) + 2\sin(\sqrt{x}) + C}$

3) $\int \sin(\ln(x))\,dx \qquad u = \sin(\ln(x)) \quad dv = dx$

$du = \cos(\ln(x))\cdot\frac{1}{x} \qquad v = x$

$= (\sin(\ln(x)))\cdot x - \int x\cdot\left(\cos(\ln(x))\cdot\frac{1}{x}\right)dx$

$= x\sin(\ln(x)) - \int \cos(\ln(x))\,dx \qquad u = \cos(\ln(x)) \quad dv = dx$

$du = -\sin(\ln(x))\cdot\frac{1}{x}\,dx \qquad v = x$

$= x\sin(\ln(x)) - \left(\cos(\ln(x))\cdot x - \int x\cdot\left(-\sin(\ln(x))\cdot\frac{1}{x}\right)dx\right)$

$= x\sin(\ln(x)) - x\cos(\ln(x)) - \int \sin(\ln(x))$

So $2\int \sin(\ln(x))\,dx = x\sin(\ln(x)) - x\cos(\ln(x))$

$\int \sin(\ln(x))\,dx = \underline{\frac{1}{2}\left(x\sin(\ln(x)) - x\cos(\ln(x))\right) + C}$

(3) a) $\int \sin(12x)\sin(7x)\,dx$

$$= \int \tfrac{1}{2}\left(\cos((12-7)x) - \cos((12+7)x)\right) dx$$

$$= \frac{1}{2}\int \cos(5x) - \cos(19x)\,dx$$

$$= \underline{\underline{\frac{1}{2}\left(\frac{1}{5}\sin(5x) - \frac{1}{19}\cos(19x)\right) + C}}$$

b) For $n \neq m$,

$$\frac{2}{\pi}\int_0^\pi \sin(mx)\sin(nx)\,dx = \frac{2}{\pi}\int_0^\pi \tfrac{1}{2}\left(\cos((m-n)x - \cos((m+n)x)\right) dx$$

$$= \frac{1}{\pi}\left[\frac{1}{m-n}\sin((m-n)x) - \frac{1}{m+n}\sin((m+n)x)\right]_0^\pi$$

$= \underline{\underline{0}}$, since $0 = \sin(0) = \sin(\pm\pi) = \sin(\pm 2\pi) = \sin(\pm 3\pi) = \cdots$

[Note: when $n = m$, $\frac{1}{m-n}$ is undefined, so this computation doesn't work.]

$$\frac{2}{\pi}\int_0^\pi \sin(mx)\sin(mx)\,dx = \frac{2}{\pi}\int_0^\pi \tfrac{1}{2}(1 - \cos(2mx))\,dx$$

$$= \frac{1}{\pi}\int_0^\pi 1 - \cos(2mx)\,dx = \frac{1}{\pi}\left[x - \frac{1}{2m}\sin(2mx)\Big|_0^\pi\right]$$

$$= \frac{1}{\pi}\left(\pi - \frac{1}{2m}\sin(2m\pi) - 0 + \frac{1}{2m}\sin(0)\right)$$

$$= \frac{1}{\pi}\cdot\pi = \underline{\underline{1}}$$

c) $$\frac{2}{\pi}\int_0^\pi f(x)\sin(mx)\,dx = \sum_{n=1}^{k} \frac{2}{\pi}\int_0^\pi a_n \sin(nx)\sin(mx)\,dx$$

$$= \sum_{n=1}^{k} a_n \cdot \left(\frac{2}{\pi}\int_0^\pi \sin(nx)\sin(mx)\,dx\right) = a_m$$

since all the terms in the sum are 0, except when $n = m$, in which case it's $a_m \cdot 1$.