(20) The integral Test does not apply to $\sum_{k=1}^{\infty} \frac{k}{\sqrt{k^2+4}}$, since the function $f(x) = \frac{x}{\sqrt{x^2+4}}$ is not decreasing on $[1, \infty)$. Since $f(x) < 1$ on $[1,\infty)$ and $f(x) \to 1$ as $x \to \infty$, it cannot be decreasing. Alternatively, we can look at the derivative:

$$\frac{d}{dx}\left(\frac{x}{\sqrt{x^2+4}}\right) = \frac{\sqrt{x^2+4}\cdot 1 - x\cdot\left(\frac{1}{2\sqrt{x^2+4}}\cdot 2x\right)}{\left(\sqrt{x^2+4}\right)^2} = \frac{(x^2+4)-x^2}{\left(\sqrt{x^2+4}\right)^3} = \frac{4}{\left(\sqrt{x^2+4}\right)^3}.$$

Since the derivative is positive, $f(x)$ is increasing.

(26) The integral Test applies to $\sum_{k=3}^{\infty} \frac{1}{k(\ln k)(\ln\ln k)}$ because the function $\frac{1}{x(\ln x)(\ln\ln x)}$ is decreasing on $[3, \infty)$. [x, $\ln(x)$, and $\ln\ln x$ are all increasing.]

$$\int \frac{1}{x(\ln x)(\ln\ln x)}\,dx = \int \frac{1}{w}\,dw = \ln\ln\ln x + C,$$

$$\left(w = \ln\ln x,\ dw = \frac{1}{x\ln x}\,dx\right)$$

$$\text{So } \int_3^{\infty} \frac{1}{x(\ln x)(\ln\ln x)}\,dx = \lim_{b\to\infty} \ln\ln\ln x \Big|_3^b = \infty$$

By the integral Test, $\sum_{k=3}^{\infty} \frac{1}{k(\ln k)(\ln\ln k)}$ also diverges.

(34) $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{27k^2}} = \sum_{k=1}^{\infty} \frac{1}{3\sqrt[3]{k^2}} = \frac{1}{3}\sum_{k=1}^{\infty} \frac{1}{k^{2/3}}$. $\sum_{k=1}^{\infty} \frac{1}{k^{2/3}}$ is a p-series with $p = \frac{2}{3} < 1$. So $\sum_{k=1}^{\infty} \frac{1}{k^{2/3}}$ diverges, and $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{27k^2}}$ also diverges.

(36) a) The remainder when estimating $\sum_{k=1}^{\infty} \frac{1}{k^8}$ as $\sum_{k=1}^{n} \frac{1}{k^8}$ is $\sum_{k=n+1}^{\infty} \frac{1}{k^8}$, which is $< \int_n^{\infty} \frac{1}{x^8}\,dx$ [formula (2), p. 634]

$\int_n^{\infty} \frac{1}{x^8}\,dx = -\frac{1}{7x^7}\Big|_n^{\infty} = \frac{1}{7n^7}$, so the remainder is $< \frac{1}{7n^7}$

b) To find out how large n has to be to ensure this remainder is less than 10^{-3}, try some values of n...

For $n=2$, $\frac{1}{7n^7} \approx 0.00116$, and for $n=3$, $\frac{1}{7n^7} \approx 0.000065$

so 3 terms are enough for an error < 0.001.

c) $\int_1^\infty \frac{1}{x^8}\,dx < \sum_{k=1}^{\infty} \frac{1}{k^8} < \frac{1}{1^8} + \int_1^\infty \frac{1}{x^8}\,dx$

Since $\int_1^\infty \frac{1}{x^8}\,dx = -\frac{1}{7x^7}\Big|_1^\infty = \frac{1}{7}$, we must have

$\frac{1}{7} \le \sum_{k=1}^{\infty} \frac{1}{k^8} \le 1 + \frac{1}{7}$ [I misread the question — should be $L_n = \sum_{k=1}^{n} \frac{1}{k^8} + \int_{n+1}^\infty \frac{1}{x^8}\,dx$, $U_n = \sum_{k=1}^{n} \frac{1}{k^8} + \int_n^\infty \frac{1}{x^8}\,dx$ ...]

d) The error in approximating $\sum_{k=1}^{\infty} \frac{1}{k^8}$ using 10 terms is $\sum_{k=11}^{\infty} \frac{1}{k^8}$, which satisfies $\int_{11}^\infty \frac{1}{x^8}\,dx < \sum_{k=11}^{\infty} \frac{1}{k^8} < \int_{10}^\infty \frac{1}{x^8}\,dx$,

or $\frac{1}{7\cdot 11^7} < \sum_{k=11}^{\infty} \frac{1}{k^8} < \frac{1}{7\cdot 10^7}$. Add $\sum_{k=1}^{10} \frac{1}{k^8}$ to this to get

$$\left(\sum_{k=1}^{10} \frac{1}{k^8}\right) + \frac{1}{7\cdot 11^7} < \sum_{k=1}^{\infty} \frac{1}{k^8} < \left(\sum_{k=1}^{10} \frac{1}{k^8}\right) + \frac{1}{7\cdot 10^7}$$

[Approximately $1.004077358609 < \sum_{k=1}^{\infty} \frac{1}{k^8} < 1.0040773605498$]

48) $$\sum_{k=0}^{\infty} \left(\frac{1}{2}(0.2)^k + \frac{3}{2}(0.8)^k\right) = \frac{1}{2}\left(\sum_{k=0}^{\infty} (0.2)^k\right) + \frac{3}{2}\left(\sum_{k=0}^{\infty} (0.8)^k\right)$$

$$= \frac{1}{2}\left(\frac{1}{1-1/5}\right) + \frac{3}{2}\left(\frac{1}{1-4/5}\right) = \frac{1}{2}\cdot\frac{5}{4} + \frac{3}{2}\cdot\frac{5}{1} = \frac{5}{8} + \frac{15}{2} = \frac{65}{8}$$

[geometric series, $a=1$, $r=0.2=1/5$] [geometric series, $a=1$, $r=0.8=4/5$]

52) $\sum_{k=1}^{\infty} \sqrt{\frac{k+1}{k}}$ diverges by the Divergence Test, since

$$\lim_{k\to\infty} \sqrt{\frac{k+1}{k}} = \sqrt{\lim_{k\to\infty} \frac{k+1}{k}} = \sqrt{1} = 1 \ne 0.$$