Math 131-03, Spring 2017 Sample Answers to Homework #11

Section 8.5

(10) $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{2^{k+1}}{(k+1)!} \Big/ \frac{2^k}{k!} = \lim_{k\to\infty} \frac{2^{k+1}}{(k+1)!} \cdot \frac{k!}{2^k} = \lim_{k\to\infty} \frac{2}{k+1} = 0$

Since the limit exists and is less than 1, the series $\sum_{k=1}^{\infty} \frac{2^k}{k!}$ converges by the Ratio Test

(16) $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{(k+1)^6}{(k+1)!} \Big/ \frac{k^6}{k!} = \lim_{k\to\infty} \frac{(k+1)^6}{(k+1)!} \cdot \frac{k!}{k^6} = \lim_{k\to\infty} \frac{(k+1)^6}{(k+1)k^6}$

$= \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^6 \cdot \frac{1}{(k+1)} = \left(\lim_{k\to\infty} \frac{k+1}{k}\right)^6 \cdot \lim_{k\to\infty} \frac{1}{(k+1)} = 1 \cdot 0 = 0$

Since the limit exists and is less than 1, the series $\sum_{k=1}^{\infty} \frac{k^6}{k!}$ converges by the ratio test.

(28) $\lim_{k\to\infty} \frac{\left(\frac{k^2+k-1}{k^4+4k^2-3}\right)}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{k^2+k-1}{k^4+4k-3} \cdot \frac{k^2}{1} = \lim_{k\to\infty} \frac{k^4+k^3-k^2}{k^4+4k-3} = 1$

[highest power rule]

Since this limit exists and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges [as a p-series with $p>1$], the series $\sum_{k=1}^{\infty} \frac{k^2+k-1}{k^4+4k^2-3}$ also converges by the Limit Comparison Test

(38) Since $\ln(k) > 1$ for $k > 2$, $k \ln k > k$, $(k \ln k)^2 > k^2$, and $\frac{1}{(k \ln k)^2} < \frac{1}{k^2}$ for $k > 2$. Since $\sum_{k=2}^{\infty} \frac{1}{k^2}$ converges, the series $\sum_{k=2}^{\infty} \frac{1}{(k \ln k)^2}$ also converges by the Comparison Test

(46) $\lim_{k\to\infty} \frac{\frac{2^k}{e^k-1}}{\left(\frac{2}{e}\right)^k} = \lim_{k\to\infty} \frac{2^k}{e^k-1} \cdot \frac{e^k}{2^k} = \lim_{k\to\infty} \frac{e^k}{e^k-1} = \lim_{k\to\infty} \frac{1}{1-e^{-k}} = 1.$

Since this limit exists, and $\sum_{k=1}^{\infty} \left(\frac{2}{e}\right)^k$ converges [as a geometric series with $r = \frac{2}{e} < 1$], then $\sum_{k=1}^{\infty} \frac{2^k}{e^k-1}$ also converges by the Limit Comparison Test.

(18) $\frac{\ln k}{k^2}$ is a positive, decreasing sequence. $\left[\frac{d}{dk}\frac{\ln k}{k^2} = \frac{k^2\cdot\frac{1}{k} - (\ln k)\cdot 2k}{(k^2)^2}\right.$

$= \frac{k - 2k\ln k}{k^4} = \frac{1-2\ln k}{k^3}$, Since $2\ln k > 1$ for $k \geq 1$, This derivative is negative. This proves that the sequence is decreasing$]$

Also, $\lim_{k\to\infty}\frac{\ln k}{k^2} = 0$. So $\sum_{k=1}^{\infty}(-1)^k\frac{\ln k}{k^2}$ converges by

The Alternating Series Test.

(30) The remainder in a convergent alternating series satisfies $R_n < |a_{n+1}|$, so we just need to choose n so that $\frac{1}{(n+1)!} < 10^{-4}$, or

$(n+1)! > 10^4$, or $(n+1)! > 10{,}000$. Since $7! = 5040$, and $8! = 40320$, $\underline{\underline{n=7}}$ works.

(34) We just need to choose n so that $\frac{1}{(2(n+1)+1)^3} < 10^{-4}$, or

$(2n+3)^3 > 10{,}000$, or $2n+3 > 21.5$, or $n > \frac{21.5-3}{2}$, so $\underline{\underline{n=10}}$ works

(46) $\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{k}}$ converges by the Alternating Series Test, since

$\left\{\frac{1}{\sqrt{k}}\right\}_{k=1}^{\infty}$ is a decreasing, positive sequence and $\lim_{k\to\infty}\frac{1}{\sqrt{k}}$ converges.

However, $\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}$ is divergent as a p-series with $p = \frac{1}{2} < 1$.

So by definition, $\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{k}}$ is <u>conditionally convergent</u>.

(50) $\sum_{k=1}^{\infty}\frac{(-1)^k e^k}{(k+1)!}$ is <u>absolutely convergent</u> since $\sum_{k=1}^{\infty}\frac{e^k}{(k+1)!}$

converges by the ratio Test. $\left[\lim_{k\to\infty}\frac{e^{k+1}}{(k+2)!}\Big/\frac{e^k}{(k+1)!}\right.$

$= \lim_{k\to\infty}\frac{e^{k+1}}{(k+2)!}\cdot\frac{(k+1)!}{e^k} = \lim_{k\to\infty}\frac{e}{k+2} = 0$, Since this

limit is < 1, $\sum_{k=1}^{\infty}\frac{e^k}{(k+1)!}$ converges.$\left.\right]$