Your work should be submitted through Canvas by 11:59 PM on Saturday, May 9.

About the exam: This exam counts for 23% of the total grade for the course. Your final project, which is due on the same day as the exam, counts for 10% of the grade. The project and exam will be submitted and graded separately.

The work that you submit for this exam should be your own. You can use course materials, including the textbook, your notes, class notes and videos from online lectures, and posted solutions to homeworks. You should not use other books or material from the Internet. You can ask your professor questions about the exam, but you should not receive help on the exam from other students, your friends and family, or anyone else.

For the problems on this exam, you should you should present neatly written solutions. Write out your answers carefully, including explanations to justify your work when appropriate.

For the essay questions on this exam, you should write out clear and well-organized responses in complete sentences and paragraphs. The questions are meant to give you an opportunity to display your understanding of the central ideas from the course. Please type your answers to essay questions in a word processing application, if at all possible, and save your work as a PDF or as a Microsoft Word or Open Office document.

As usual, you can submit your work in Canvas in the form of PDFs, image files, and/or word processing documents.

You do not need to work with complex numbers or complex vector spaces on this exam. You can assume that all of the vector spaces are real vector spaces, and all eigenvalues are real eigenvalues.

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1. Calculations. (25 points)

a) Find all solutions of the homogeneous system:

$$\begin{array}{rcrcrcrcl} x & + & 2y & & & - & w & = & 0 \\ 2x & + & 5y & - & 6z & + & 2w & = & 0 \\ & & 2y & + & 2z & + & w & = & 0 \\ -2x & - & y & + & 4z & + & 3w & = & 0 \end{array}$$

b) Is $\left\langle \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 4 \\ 7 \\ -3 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} \right\rangle$ a basis of R^3?

c) Let $\mathscr{P}_3$ be the vector space of polynomials of degree less than or equal to 3. Let $B = \langle x^3 - x^2, x^2 - x, x - 1, 1 \rangle$, which is a basis of $\mathscr{P}_3$. Find $Rep_B(4x^3 + 3x^2 + 2x + 1)$.

d) Find the determinant: $\begin{vmatrix} 0 & 3 & 0 & 0 \\ 1 & 4 & 2 & 5 \\ 3 & 3 & 3 & 0 \\ 5 & 2 & 6 & 0 \end{vmatrix}$

2. Proofs. (25 points)

a) Suppose $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ sre vectors in $\mathbb{R}^k$ and that their dot products have the property that $\vec{v}_i \cdot \vec{v}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$. (The vectors are said to be *orthonormal*; each vector has length 1 and is perpendicular to every other vector in the list.) Show that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are linearly independent. Hint: What is $\vec{v}_i \cdot (c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k)$?

b) Let V and W be vector spaces. Let $B = \langle \vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n \rangle$ be a basis V. And let $h\colon V \to W$ be a homomomorhism. Suppose that h is injective (that is, one-to-one). Show that the vectors $h(\vec{b}_1), h(\vec{b}_2), \ldots, h(\vec{b}_n)$ are linearly independent.

c) Suppose that A is an $n \times n$ matrix that has n **distinct** eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Show that if $|\lambda_i| < 1$ for $i = 1, 2, \ldots, n$, then the determinant satisfies $|\det(A)| < 1$. (Note that the notation $|\cdot|$ represents absolute value, not determinant.)

3. Essay Questions. (50 points) Please type your answers in a word processing application, if possible, and submit either a PDF or a Microsoft Word or Open Office document. Remember to write your responses as well-organized essays in full sentences and paragraphs. Each question counts for 25 points. I do not expect more than a page or so for each essay.

a) We began the course with *linear systems* and how to solve them using *row reduction*. Discuss some of the ways in which these concepts have been useful in other parts of the course, such as dealing with linear independence, matrix inverses, and determinants.

b) A large part of the course was about matrices and the vector spaces $\mathbb{R}^n$. However, we also worked with general vector spaces, V. In many cases, we worked with general finite-dimensional vector spaces by choosing bases for the vector spaces. Explain what it means to "choose a basis" and how doing so relates general vector spaces to the vector spaces $\mathbb{R}^n$ and relates homomorphisms between general vector spaces to matrices.