This homework is due on Friday, March 13.

1. Find the rank of each matrix. Fully explain your reasoning.

a) $\begin{pmatrix} -1 & 1 & 4 & 2 \\ 2 & -1 & 3 & 0 \\ 3 & 0 & -3 & 1 \end{pmatrix}$ **b)** $\begin{pmatrix} 1 & 3 & -2 & 1 \\ 2 & -1 & 1 & 3 \\ 3 & 2 & -1 & 4 \\ 1 & -4 & 3 & 2 \end{pmatrix}$

2. Suppose the matrix $\begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & x \\ 2 & 1 & x \end{pmatrix}$ has rank 2. What is x? Explain your reasoning.

3. Let $B = \{\vec{\beta}_1, \vec{\beta}_2, \dots \vec{\beta}_n\}$ be a basis for some vector space V. Let $\vec{v} \in V$. We know from the Exchange Lemma that if $\vec{v} = c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n$, and if $c_i \neq 0$, then $\vec{v}$ can be exchanged for $\vec{\beta}_i$ in B to produce a new basis $\hat{B} = \{\vec{\beta}_1, \vec{\beta}_2, \dots, \vec{\beta}_{i-1}, \vec{v}, \vec{\beta}_{i+1}, \dots, \vec{\beta}_n\}$.

a) Explain why if $\vec{v}$ is any non-zero vector in V, then $\vec{v}$ can be exchanged for at least one vector in the basis B.

b) Can it happen that for some non-zero vector $\vec{v} \in V$, there is **exactly one** vector in $\vec{v}$ for which $\vec{v}$ can be exchanged? If not, why not? If so, what has to be true about $\vec{v}$?

4. Consider a general 3×3 matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$. To keep the notation manageable, this problem looks at matrix multiplication of vectors by A, but it could be generalized to apply to any matrix.

a) We have seen how to multiply a vector by a matrix, and we have seen that multiplication by an $m \times n$ matrix defines a function from $\mathbb{R}^n$ to $\mathbb{R}^m$. This problem asks you to verify for the case of a 3×3 matrix that that function respects vector operations. That is, show that for any $\vec{v} \in \mathbb{R}^3$ and $r \in \mathbb{R}$, $A(r\vec{v}) = r \cdot (A\vec{v})$ and for any $\vec{v}, \vec{w}$, $A(\vec{v} + \vec{w}) = (A\vec{v}) + (A\vec{w})$. (This is an easy calculation.)

b) Let $\vec{e}_1, \vec{e}_2, \vec{e}_3$ be the usual standard basis for $\mathbb{R}^3$. Show that $A\vec{e}_i = a_i$ for $i = 1, 2, 3$, where a_i is the i-th column of A. Again, this is an easy calculation.

c) Deduce that the function from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined as multiplication by the matrix A is completely determined by its effect on the basis vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$. That is, once you know $A\vec{e}_1, A\vec{e}_2, A\vec{e}_3$, you know what $A\vec{v}$ must be for any $\vec{v} \in \mathbb{R}^3$.

5. Let V and W be vector spaces. Suppose that $f\colon V \to W$ is a linear function. (That is, $f(r \cdot \vec{v}) = r \cdot f(\vec{v})$ for all $r \in R$, $\vec{v} \in V$, and $f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2)$ for all $\vec{v}_1, \vec{v}_2 \in V$. let $K = f^{-1}(\vec{0})$. That is, $K = \{\vec{v} \in V \mid f(\vec{v}) = \vec{0}\}$. Show that K is a subspace of V.

6. Let $\mathcal{P}_n$ be the vector space of polynomials of degree less than or equal to N, as usual. Show that the function $f\colon \mathbb{R}^3 \to \mathcal{P}_2$ given by $f\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (a+b+c)+(b+c)x+cx^2$ is an isomorphism of vector spaces.

7. Show that the function $f\colon \mathbb{R}^4 \to \mathbb{R}^4$ given by $f\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} d \\ c \\ b \\ a \end{pmatrix}$ is an automorphism.

8. Let $\langle \vec{b}_1, \vec{b}_2 \rangle$ be any basis of $\mathbb{R}^2$. Define a function $f\colon \mathbb{R}^2 \to \mathbb{R}2$ by $f\begin{pmatrix} x \\ y \end{pmatrix} = x\vec{b}_1 + y\vec{b}_2$. Show that f is an automorphism.