This homework is due on Monday, April 20.

1. Let $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the homomorphism $h\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x-y\\y-z\\z-x\end{pmatrix}$.

a) Find the kernel of h and show that it has dimension 1.

b) Since the nullity of h is 1, there are bases B and D of $\mathbb{R}^3$ such that $Rep_{B,D}(h) = \begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$. Find bases B and D with this property. (Recall that B will include a basis of the kernel of h. There are infinitely many correct answers to this question!)

c) Find the matrix $Rep_{B,D}(h)$ if $B = \left\langle \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right\rangle$ and D is the standard basis.

d) Find the matrix $Rep_{B,D}(h)$ if $B = \left\langle \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right\rangle$ and $D = \left\langle \begin{pmatrix}-1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}1\\3\\-2\end{pmatrix} \right\rangle$.

(Part d is harder and messier.)

2. Find the determinants of the following matrices. Most of them are very easy, using properties of determinants! When you use a property of determinants in your computation, state the property.

a) $\begin{vmatrix}3&-1\\2&5\end{vmatrix}$ **b)** $\begin{vmatrix}2&3&4\\0&3&4\\0&0&4\end{vmatrix}$ **c)** $\begin{vmatrix}3&-1&1\\-3&-1&0\\6&-4&7\end{vmatrix}$

d) $\begin{vmatrix}0&0&2\\0&5&3\\6&-2&4\end{vmatrix}$ **e)** $\begin{vmatrix}1&3&7&5\\2&3&8&1\\1&3&7&5\\1&2&3&4\end{vmatrix}$ **f)** $\begin{vmatrix}1&2&3\\-2&3&5\\-1&5&2\end{vmatrix}$

3. For a finite-dimensional vector space V, and a linear transformation $h\colon V \to V$, let's consider the question, does it make sense to try to define a determinant for h? To do so, we need a matrix representation for h, which means choosing a basis for V. If B is the basis, we get the matrix $Rep_{B,B}(h)$, and we can take the determinant of that matrix. But if we choose a different basis D, we get a different matrix $Rep_{D,D}(h)$. Do these two matrix representations always have the same determinant? To make this question very easy, just recall that

$$Rep_{D,D}(h) = Rep_{B,D}(id)^{-1} \cdot Rep_{B,B}(h) \cdot Rep_{B,D}(id)$$

4. The cross product of two vectors $\vec{v}, \vec{w} \in \mathbb{R}^3$ is a vector, $\vec{v} \times \vec{w}$, that is orthogonal to both $\vec{v}$ and $\vec{w}$. The cross product of $\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ and $\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$ can be remembered as the formal determinant

$\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$. Write out this determinant, using the formula for the determinant of a 3×3 matrix, and show that the result is, in fact, orthogonal to both vectors. (Recall that two vectors are orthogonal if their dot product is zero.)

5. Let A be an $n \times n$ matrix. Suppose $\vec{v}$ is a non-zero vector in $\mathbb{R}^n$ and $\lambda \in \mathbb{R}$. If $\vec{v}$ and λ the property that $A\vec{v} = \lambda \cdot \vec{v}$, then λ is called an **eigenvalue** of A, and $\vec{v}$ is an **eigenvector** or A (with eigenvalue λ).

a) Show if λ is an eigenvalue of A, then the matrix $A - \lambda I_n$ is singular, and the eigenvectors with eigenvalue λ are just the non-zero vectors $\vec{v}$ such tnat $(A - \lambda I_n)\vec{v} = 0$.

b) Show that if the matrix $A - \lambda I_n$ is singular, then λ is an eigenvalue of A.

c) Deduce that a real number λ is an eigenvalue of A if and only if $det(A - \lambda I_n) = 0$.

d) If x is a variable and A is an $n \times n$ matrix, then $p(x) = det(A - xI_n)$ is a polynomial in x. The eigenvalues of A are just the solutions of $p(x) = 0$. Use this fact to find the eigenvalues of $A = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 3 & -1 \\ 0 & 0 & -1 \end{pmatrix}$, and then find an eigenvector for each eigenvalue. Note that $|A - xI_n| = \begin{vmatrix} 2-x & 0 & 1 \\ -1 & 3-x & -1 \\ 0 & 0 & -1-x \end{vmatrix}$, and this detminant is very easy to compute using the formula for the determinant of a 3×3 matrix.