Metric Spaces

Math 331, Handout #1

We have looked at the "metric properties" of $\mathbb{R}$: the distance between two real numbers x and y is defined to be $|x - y|$. We can generalize this idea to define "metric spaces." A metric space is a set together with a notion of distance between any two elements of that set. The properties that must be satisfied by distance in a metric space are modeled on the properties of distance in $\mathbb{R}$.

Definition 1. A **metric space** is a pair (M, d) where M is a set and d is a function $d\colon M \times M \to \mathbb{R}$ satisfying

1. $d(x, y) \geq 0$, for all $x, y \in M$, and $d(x, y) = 0$ if and only if $x = y$;
2. symmetry: $d(x, y) = d(y, x)$, for all $x, y \in M$; and
3. the triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in M$.

A function d satisfying the above properties is said to be a **metric** on M.

In $\mathbb{R}$, we have the idea of an **open set**. A subset of $\mathbb{R}$ is open in $\mathbb{R}$ if it is a union of open intervals. Another way to define an open set is in terms of distance. A set is open in $\mathbb{R}$ if whenever it contains a number a, it also contains all numbers "sufficiently close" to a. More precisely, A subset A of $\mathbb{R}$ is open in R if for every $a \in A$, there is an $\epsilon > 0$ such that the open interval $(a - \epsilon, a + \epsilon)$ is a subset of A. The open interval $(a - \epsilon, a + \epsilon)$ consists of all numbers that are within distance ϵ of a.

In a general metric space, the analog of the interval $(a - \epsilon, a + \epsilon)$ is the "open ball of radius ϵ about a," and we can define a set to be open in a metric space if whenever it includes a point a, it also includes an entire open ball of radius epsilon about a.

Definition 2. Let (M, d) be a metric space. For $a \in M$ and $\epsilon > 0$, we define the **open ball about** a **of radius** ϵ to be the set $B_\epsilon(a) = \{x \in M \mid d(x, a) < \epsilon\}$. A subset X of M is said to be **open in** M if and only if for every $a \in X$, there is an $\epsilon > 0$ such that $B_\epsilon(a) \subseteq X$.

To make it clear what metric is being used, the open ball $B_\epsilon(a)$ can also be written $B^d_\epsilon(a)$ when the metric space is not clear from context.

To justify the name "open ball," we should check that an open ball is in fact an open set according to the above definition. This fact can be proved using the triangle inequality:

Theorem 1. Let (M, d) be a metric space. Let $a \in M$ and $\epsilon > 0$. Then the open ball $B_\epsilon(a)$ is an open set.

Proof. Let b be any point in $B_\epsilon(a)$. To show $B_\epsilon(a)$ is open, we must show that there exists $\delta > 0$ such that $B_\delta(b) \subseteq B_\epsilon(a)$. Let $\delta = \frac{1}{2}\big(\epsilon - d(b, a)\big)$. Since $b \in B_\epsilon(a)$, we know by definition that

$d(b,a) < \epsilon$, so that $\delta > 0$. To show $B_\delta(b) \subseteq B_\epsilon(a)$, let $c \in B_\delta(b)$. By definition, this means that $d(c,b) < \delta$. By the triangle inequality, we then have

$$\begin{aligned} d(c,a) &< d(c,b) + d(b,a) \\ &< \delta + d(b,a) \\ &= \tfrac{1}{2}\big(\epsilon - d(b,a)\big) + d(b,a) \\ &= \tfrac{1}{2}\epsilon - \tfrac{1}{2}d(b,a) + d(b,a) \\ &= \tfrac{1}{2}\epsilon + \tfrac{1}{2}d(b,a) \\ &< \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon, \qquad \text{since } d(b,a) < \epsilon \\ &= \epsilon \end{aligned}$$

This shows that $c \in B_\epsilon(a)$, and since c is any point in $B_\delta(b)$, we have shown that $B_\delta(b) \subseteq B_\epsilon(a)$. □

A major example of metric space is $(\mathbb{R}^n, d)$, where d is the usual distance measure on $\mathbb{R}^n$, which is given by

$$d\big((x_1, \ldots, x_n),\ (y_1, \ldots, y_n)\big) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

Note that for $n = 1$, this metric reduces to $d(x,y) = |x - y|$. In $(\mathbb{R}^n, d)$, the open ball $B_\epsilon(\vec{x})$ is the n-dimensional sphere of radius ϵ centered at $\vec{x}$.

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The open sets in a metric space (M,d) define a "topology" on M and make M into a "topological space." We will not cover topological spaces here, but the following theorem says that the collection of open sets in a metric space satisfy the axioms for a topology. The proof is left as an exercise.

Theorem 2. Let (M,d) be a metric space. Then the open sets in M satisfy the following properties:

1. M is open and $\varnothing$ is open;
2. the union of any collection (finite or infinite) of open sets is open; and
3. the intersection of any finite collection of open sets is open.

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We can also define **closed** sets in a metric space. A subset C of a metric space (M,d) is said to be closed if and only if its complement, $\overline{M} = M \smallsetminus C$, is open. Note that a set can be both open and closed; for example, the empty set is both open and closed. By applying DeMorgan's laws to the previous theorem, we immediately get a similar theorem for closed sets:

Theorem 3. Let (M,d) be a metric space. Then the closed sets in M satisfy the following properties:

1. M is closed and $\varnothing$ is closed;
2. the intersection of any collection (finite or infinite) of closed sets is closed; and
3. the union of any finite collection of closed sets is closed.

We can give an alternative definition of closed set using the idea of "accumulation point" (or "cluster point" or "limit point"). An accumulation point of a set is a point that is arbitrarily close to elements of that set.

Definition 3. Let (M, d) be a metric space, let $X \subseteq M$, and let $x \in X$. Then x is a **accumulation point** of X if and only if for every $\epsilon > 0$, $X \cap (B_\epsilon(x) \smallsetminus \{x\}) \neq \varnothing$. (That is, for any $\epsilon > 0$, there is at least one element of X, other than x itself, that is within distance ϵ of x.)

Note that an accumulation point of X might or might not be an element of X. For example, consider the metric space $(\mathbb{R}, d)$, where d is the usual metric. The accumulation points of the open interval $X = (0, 1)$ are all points in the closed interval $[0, 1]$. In this example, every point of X is an accumulation point, and there are two additional accumulation points of X, zero and one, that are not elements of X. The only accumulation point of the set $Y = \{\frac{1}{n} \,|\, n \in \mathbb{N}\}$ is the number zero, so in this case, no accumulation point of Y is an element of Y. For any closed set X, every accumulation point of X is an element of X. In fact, this property characterizes closed sets.

Theorem 4. Let (M, d) be a metric space, and let $X \subseteq M$. X is closed if and only if every accumulation point of X is an element of X.

Proof. Suppose first that X is closed. Let x be an accumulation point of X. We need to show that $x \in X$. Suppose, for that sake of contradiction, that $x \notin X$. Then x is in the complement, $M \smallsetminus X$. Since X is closed, we know by definition of closed that $M \smallsetminus X$ is open. Since $M \smallsetminus X$ is open and $x \in M \smallsetminus X$, we know by definition of open that there is an $\epsilon > 0$ such that $B_\epsilon(x) \subseteq M \smallsetminus X$. Since $B_\epsilon(x)$ is entirely contained in the complement of X, the intersection $X \cap B_\epsilon(x) = \varnothing$. But this contradicts the fact that x is an accumulation point of X. This contradiction shows that $x \notin X$ is impossible, and so we must have $x \in X$.

Conversely, suppose that every accumulation point of X is an element of X. We need to show that X is closed. That is, we must show that the complement, $M \smallsetminus X$, is open. Let $a \in M \smallsetminus X$. By definition of open, we need to find an $\epsilon > 0$ such that $B_\epsilon(a) \subseteq M \smallsetminus X$. Since $a \notin X$ and every accumulation point of X is in X, it follows that a is not an accumulation point of X. By definition of accumulation point, this means that there is an $\epsilon > 0$ such that $X \cap (B_\epsilon(a) \smallsetminus \{a\}) = \varnothing$. Since we also know $a \notin X$, we have in fact $X \cap B_\epsilon(a) = \varnothing$, which is equivalent to $B_\epsilon(a) \subseteq M \smallsetminus X$. □

Definition 4. Let (M, d) be a metric space, and let X be a subset of M. We define $\overline{X}$, the **closure** of X, to be the set consisting of all the points of X together with all the accumulation points of X.

Theorem 5. Let (M, d) be a metric space, and let X be a subset of M. Then $\overline{X}$ is closed.

Proof. By the previous theorem, it suffices to show that $\overline{X}$ contains all of its accumulation points. So, suppose that x is an accumulation point of $\overline{X}$. We need to show $x \in \overline{X}$, that is, we need to show that either $x \in X$ or or x is an accumulation point of X. In fact, we show that x is an accumulation point of X.

Let $\epsilon > 0$. We need to show that $X \cap (B_\epsilon(x) \smallsetminus \{x\}) \neq \varnothing$. Since x is an accumulation point of $\overline{X}$, we know that $\overline{X} \cap (B_{\epsilon/2} \smallsetminus \{x\}) \neq \varnothing$. Let $y \in \overline{X} \cap (B_{\epsilon/2} \smallsetminus \{x\})$. Since $y \in \overline{X}$, either $y \in X$ or y is an accumulation point of X. If y is in X, then y is in $X \cap (B_\epsilon(x) \smallsetminus \{x\})$ and we are done. Suppose y is an accumulation point of X. Let $\delta = d(x, y)$, which is greater than zero and less than $\frac{\epsilon}{2}$. We then have $X \cap (B_\delta(y) \smallsetminus \{y\}) \neq \varnothing$. Let $z \in X \cap (B_\delta(y) \smallsetminus \{y\})$. Then $d(z, x) \leq d(z, y) + d(y, x) < \delta + d(y, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. That is, $z \in B_\epsilon(x)$. Furthermore, $z \neq x$ (since $d(z, y) < d(x, y)$), and $z \in X$. That is, $z \in X \cap (B_\epsilon(x) \smallsetminus \{x\})$, so that we are done in this case as well. □

Note that if C is a closed subset of a metric space, then the fact that C already contains all of its accumulation points means that $\overline{C} = C$. And if X is any subset of a metric space, the fact that $\overline{X}$ is closed means that $\overline{\overline{X}} = \overline{X}$.

Exercises

Exercise 1. Let X be any set, and define $\rho\colon X \times X \to \mathbb{R}$ by $\rho(a,b) = \begin{cases} 0 & \text{if } a = b \\ 1 & \text{if } a \neq b \end{cases}$. Show that ρ is a metric for X. What are the open sets and the closed sets in the metric space (X, ρ)? (Hint: What is $B_{1/2}(x)$?) The metric ρ is called the **discrete metric** on X.

Exercise 2. Prove Theorem 2.

Exercise 3. Find an infinite collection of open subsets of $(\mathbb{R}, d)$ whose intersection is not open. Find an infinite collection of closed subsets of $(\mathbb{R}, d)$ whose union is not closed.

Exercise 4. Consider the set $C([a,b])$, the set of continuous functions on the closed, bounded interval $[a,b]$. Define a metric d on this set by

$$d(f,g) = \int_a^b |f(x) - g(x)|\, dx$$

Show that d is in fact a metric. For a given $f \in C([a,b])$ and $\epsilon > 0$, try to describe what it means for a function g to be in $B^d_\epsilon(f)$. (Draw some pictures!)

Exercise 5. Consider the metric space $\mathbb{R}$ (with its usual metric). Show that 0 is an accumulation point of the set $X = \{\frac{1}{n} \,|\, n \in \mathbb{N}\}$, and that it is the only accumulation point.

Exercise 6. Suppose X is a subset of a metric space (M, d) and that x is an accumulation point of X. Let $\epsilon > 0$. We know that $X \cap (B_\epsilon(x) \smallsetminus \{x\})$ non-empty. Show that in fact, $X \cap (B_\epsilon(x) \smallsetminus \{x\})$ is infinite. (Hint: Suppose that it is finite and show how to find another element of $X \cap (B_\epsilon(x) \smallsetminus \{x\})$ that is closer to x than any of those finitely many points.)

Exercise 7. Show that any finite subset of a metric space is a closed subset of that metric space.

Exercise 8. Draw a picture, using $\mathbb{R}^2$ with its usual metric, to illustrate the proof of Theorem 1. Explain!

Exercise 9. Suppose that A is a closed, bounded subset of $\mathbb{R}$, and let $\lambda = \text{lub}(A)$. Show that $\lambda \in A$. (Similarly, A contains its greatest lower bound.)