Compactness in Metric Spaces

Math 331, Handout #2

We have proved the Heine-Borel Theorem for closed bounded intervals in $\mathbb{R}$: If $[a,b]$ is a closed bounded interval, then every open cover of $[a,b]$ has a finite subcover. This property can be extended to certain other subsets of $\mathbb{R}$ and to certain subsets of metric spaces in general. The sets in question are said to be "compact." We can take the finite subcover property to be the definition of compactness, although we will see that there are other equivalent properties that also characterize compactness. Note that compactness can be defined entirely in terms of open sets, without mentioning the distance measure. This means that compactness is really a topological property rather than a metric property.

Definition 1. Let (M,d) be a metric space. A subset, K, of M is said to be **compact** if and only if every open cover of K (by open sets in M) has a finite subcover. If M itself has this property, then we say that M is a compact metric space.

We start with the fact that in any metric space, a compact subset is closed and bounded. Bounded here means that the subset "does not extend to infinity," that is, that it is contained in some open ball around some point. But we can give an alternative definition, which says that a set is bounded if there is a limit on how far apart two points in the set can be. We can also define the "diameter" of such a subset:

Definition 2. Let (M,d) be a metric space. A subset X of M is said to be **bounded** if and only if the set $\{d(x,y)\,|\,x,y \in X\}$ is bounded above. For a non-empty bounded set X, we define $diam(X)$, the **diameter** of X, to be the least upper bound of the set $\{d(x,y)\,|\,x,y \in X\}$ (which exists by the least upper bound property of $\mathbb{R}$, since the set is bounded above).

Take careful note of how the finite subcover property is used in the following proof; the technique is common in proofs about compactness.

Theorem 1. Let (M,d) be a metric space, and let K be a compact subset of M. Then K is a closed subset of M, and K is bounded.

Proof. Let K be a compact subset of a metric space (M,d). To prove that K is closed, we show that the complement, $G = M \setminus K$, is open. Let $z \in G$. We need to find $\epsilon > 0$ such that $B_\epsilon(z) \subseteq G$. Now, for any $x \in K$, let $\epsilon_x = d(x,z)$. Since $z \notin K$, $\epsilon_x > 0$. The collection of open sets $\{B_{\epsilon_x}(x)\,|\,x \in K\}$ is an open cover of K (since any $x \in K$ is covered by $B_{\epsilon_x}(x)$). Since K is compact, there is a finite subcover of this cover; that is, there is a finite set $x_1, x_2, \dots, x_n$ such that the corresponding open balls already cover K. Let $\epsilon = \frac{1}{2}\min(\epsilon_{x_1}, \epsilon_{x_2}, \dots, \epsilon_{x_n})$. The claim is that $B_\epsilon(z) \subseteq G$. To show this, let $y \in B_\epsilon(z)$. We want to show $y \in G$, that is, $y \notin K$. Consider x_i, where $1 \le i \le n$. By the triangle inequality, $d(x_i,y) + d(y,z) \ge d(x_i,z) = \epsilon_{x_i} \ge 2\epsilon$. So, $d(x_i,y) \ge 2\epsilon - d(y,z) > 2\epsilon - \epsilon = \epsilon$. (The last inequality follows because $d(y,z) < \epsilon$.) Then, since $d(x_i,y) > \epsilon$, y is not in the open ball of radius ϵ_{x_i} about x_i. Since the open balls $B_{\epsilon_{x_i}}(x_i)$ cover K, we have that $y \notin K$.

To prove that K is bounded, let x be some element of K, and consider the collection of open balls of integral radius, $\{B_i(x) \,|\, i = 1, 2, \dots\}$. Since every element of K has some finite distance from x, this collection is an open cover of K, Since K is compact, it has a finite subcover $\{B_{i_1}(x), B_{i_2}(x), \dots, B_{i_n}(x)\}$, where we can assume $i_1 < i_2 < \cdots < i_n$. But since $B_{i_1}(x) \subseteq B_{i_2}(x) \subseteq \cdots \subseteq B_{i_n}(x)$, this means that B_{i_n} by itself already covers K. Then, for $y, z \in K$, y and z are in $B_{x_n}(x)$, and $d(y, z) \le d(y, x) + d(x, z) \le i_n + i_n$. It follows that $2i_n$ is an upper bound for $\{d(y, z) \,|\, y, z \in K\}$. So, K is bounded. □

It is not true in general that every closed, bounded set is compact. However, that **is** true in $\mathbb{R}^n$, with the usual metric, and this fact gives a complete characterization of compact subsets of $\mathbb{R}^n$. We will not prove this at this time, but Exercise 5 proves it for the case $n = 1$.

The Bolzano-Weirstrass Theorem for closed, bounded intervals in $\mathbb{R}$ says that any infinite subset of such an interval has an accumulation point in the interval. A similar result holds for a compact subset of a metric space.

Theorem 2. Let (M, d) be a metric space and let K be a compact subset of M. Then any infinite subset of K has an accumulation point in K.

Proof. We prove the contrapositive. Let $X \subseteq K$. Assume that X has no accumulation point in K. We must show that X is finite. Let $z \in K$. Since z is not an accumulation point, there is an $\epsilon_z > 0$ such that $X \cap (B_{\epsilon_z}(z) \smallsetminus \{z\}) = \varnothing$. Thus, $X \cap B_{\epsilon_z}(z)$ either is empty or is $\{z\}$. The set of open balls $\{B_{\epsilon_z}(z) \,|\, z \in K\}$ is an open cover of K. Since K is compact, there is a finite cover. That is, there are finitely many points $z_1, z_2, \dots, z_n$ such that the corresponding open balls already cover K. But each of the n open balls in that subcover contains at most one point of X, and it follows that X has n or fewer points. So we have proved that X is finite. □

In fact, a set being compact is actually equivalent to the property that every infinite subset of that set has an accumulation point in the set. We have proved one direction of this equivalence. We will not prove the other direction at this time.

Exercises

Exercise 1. Find an example of a closed and bounded set C in some metric space such that C is not compact. (Hint: Consider the metric space (X, ρ) from exercise 1 in the first handout.)

Exercise 2. A subset X of a metric space is said to be **totally bounded** if for every $\epsilon > 0$, X can be covered by a finite collection of open balls of radius ϵ. Show that every compact set is totally bounded.

Exercise 3. Let (M, d) be a metric space, let X be a non-empty subset of M, and let z be an element of M. Define $d(z, X) = glb\{d(x, z) \,|\, x \in X\}$. Note that if $z \in X$, then $d(z, X) = 0$. Find an example where $z \notin X$ but $d(z, X) = 0$. Now suppose that K is a compact subset of M. Show that $d(z, K) = 0$ if and only if $z \in K$.

Exercise 4. Let (M, d) be a metric space. Let K be a compact subset of M, and let C be a closed subset of M. Then $K \cap C$ is compact. (Hint: The set $M \smallsetminus C$ is an open set.)

Exercise 5. Show that any bounded, closed subset of $\mathbb{R}$ is compact. (Hint: Use the previous exercise and the fact that any bounded, closed interval is compact.)

Exercise 6. Let (M, d) be a metric space, and let X be a subset of M. Show that X is bounded (in the sense that it has a finite diameter) if and only if for any $z \in M$, there is a number N such that $X \subseteq B_N(z)$.