Sequences and Continuity in Metric Spaces

Math 331, Handout #3

The main purpose of this handout is to give two properties of functions that are equivalent to continuity, one in terms of sequences and one in terms of open sets. We will work with functions between general metric spaces, so we need to start by defining continuity of functions and convergence of sequences in metric spaces. The definitions are the obvious generalizations of continuity and converge for $\mathbb{R}$.

Definition 1. Let (M, d) be a metric space. Let $\{x_i\}_{n=1}^{\infty}$ be an infinite sequence of points in M. Let $z \in M$. We say that the sequence $\{x_i\}_{n=1}^{\infty}$ **converges to** z if for every $\epsilon > 0$, there is a natural number N such that $x_i \in B_\epsilon(z)$ for all $n \geq N$. In that case we write $\lim\limits_{i\to\infty} x_i = z$. We say that a sequence in M is **convergent** if it converges to some point in M.

Definition 2. Let (A, ρ) and (B, τ) be metric spaces, and let f be a function $f\colon A \to B$. Let $a \in A$. We say that f is **continuous at** a if for every $\epsilon > 0$, there is a $\delta > 0$ such that $f(B^\rho_\delta(a)) \subseteq B^\tau_\epsilon(f(a))$. (That is, for every $\epsilon > 0$, there is a $\delta > 0$ such that for all $x \in A$, if $\rho(x, a) < \delta$, then $\tau(f(x), f(a)) < \epsilon$.) For a subset X of A, we say that f is **continuous on** X if f is continuous at x for every $x \in X$. We say that f is **continuous** if f is continuous at every point of A.

Our first characterization of continuity is in terms of convergent sequences: A function is continuous if and only if it preserves convergence of sequences. More formally, we have:

Theorem 1. Let (A, ρ) and (B, τ) be metric spaces, and let f be a function $f\colon A \to B$. Let $a \in A$. Then f is continuous at a if and only if for every sequence $\{a_i\}_{i=1}^{\infty}$ in A that converges to a, the sequence $\{f(a_i)\}_{i=1}^{\infty}$ in B converges to $f(a)$.

Proof. Suppose that f is continuous at a, and let $\{a_i\}_{i=1}^{\infty}$ be a sequence in A that converges to a. We must show that $\{f(a_i)\}_{i=1}^{\infty}$ converges to $f(a)$. Let $\epsilon > 0$. We need to find N such that $f(a_i) \in B^\tau_\epsilon(f(a))$ for all $i \geq N$. Since f is continuous at a, there is a $\delta > 0$ such that $f(B^\rho_\delta(a)) \subseteq B^\tau_\epsilon(f(a))$. Since the sequence $\{a_i\}_{i=1}^{\infty}$ converges to a and $\delta > 0$, there is an N such that $a_i \in B^\rho_\delta(a)$ for all $i \geq N$. Then, for any $i \geq N$, the facts that $a_i \in B^\rho_\delta(a)$ and that $f(B^\rho_\delta(a)) \subseteq B^\tau_\epsilon(f(a))$ together show that $f(a_i) \in B^\tau_\epsilon(f(a))$, which is what we needed to prove.

To prove the converse, we prove the contrapositive. Suppose that f is **not** continuous at a. We must show that it is **not** the case that for every sequence $\{a_i\}_{i=1}^{\infty}$ in A that converges to a, it follows that $\{f(a_i)\}_{i=1}^{\infty}$ converges to $f(a)$. We do this by constructing a sequence $\{a_i\}_{i=1}^{\infty}$ in A that converges to a, but $\{f(a_i)\}_{i=1}^{\infty}$ does not converge to $f(a)$.

Since f is not continuous at a, there is an $\epsilon > 0$ such that for every $\delta > 0$, $f(B^\rho_\delta(a)) \not\subseteq B^\tau_\epsilon(f(a))$. For each $i \in \mathbb{N}$, letting $\delta = \frac{1}{i}$, we have that $f(B^\rho_{1/i}(a)) \not\subseteq B^\tau_\epsilon(f(a))$; thus, there must be some $a_i \in B^\rho_{1/i}(a)$ such that $f(a_i) \notin B^\tau_\epsilon(f(a))$. Then the sequence $\{a_i\}_{i=1}^{\infty}$ converges to a (because given any $\delta > 0$, we can choose an integer $N > \frac{1}{\delta}$, and we then have for $n \geq N$ that $\frac{1}{n} \leq \frac{1}{N} < \delta$ and therefore. $a_i \in B^\rho_{1/i}(a) \subseteq B^\rho_\delta(a)$ for all $i > N$). However, the sequence $\{f(a_i)\}_{i=1}^{\infty}$ does not converge to $f(a)$ (because for every $i \in \mathbb{N}$, $f(a_i) \notin B^\tau_\epsilon(f(a))$). $\square$

One nice application of the sequential characterization of continuity is an easy proof that the composition of continuous functions is continuous.

Theorem 2. Let (A, ρ), (B, τ), and (C, σ) be metric spaces. Let $f\colon A \to B$ and $g\colon B \to C$. Let $a \in A$. If f is continuous at a and g is continuous at $f(a)$, then the composition $g \circ f$ is continuous at a.

Proof. Let $\{a_i\}_{i=1}^{\infty}$ be any sequence in A that converges to a. Since f is continuous at a, the sequence $\{f(a_i)\}_{i=1}^{\infty}$ converges to $f(a)$. Since g is continuous at $f(a)$, the sequence $\{g(f(a_i))\}_{i=1}^{\infty}$ converges to $g(f(a))$, which is $g \circ f(a)$. Since this is true for any sequence in A that converges to a, $g \circ f$ is continuous at a. $\square$

Our second characterization of continuity is in terms of open sets. The fact that continuity can be defined without using the metrics shows that it is really a topological rather than a metric property. The theorem can be stated most cleanly in terms of continuity on an entire metric space. In the proof, we use the fact that if f is a function $f\colon A \to B$, then $f(f^{-1}(Y)) \subseteq Y$ for all $Y \subseteq B$, and $X \subseteq f^{-1}(f(X))$ for all $X \subseteq A$.

Theorem 3. Let (A, ρ) and (B, τ) be metric spaces, and let f be a function $f\colon A \to B$. Then f is continuous if and only if for every open subset $\mathcal{O}$ of B, the inverse image $f^{-1}(\mathcal{O})$ is open in A.

Proof. Suppose f is continuous, and $\mathcal{O}$ is an open subset of B. We need to show that $f^{-1}(\mathcal{O})$ is open in A. Let $a \in f^{-1}(\mathcal{O})$. We need to find an open ball about a that is contained in $f^{-1}(\mathcal{O})$. By definition of f^{-1}, we have that $f(a) \in \mathcal{O}$. Since $\mathcal{O}$ is open, there is an $\epsilon > 0$ such that $B^{\tau}_{\epsilon}(f(a)) \subseteq \mathcal{O}$. Since f is continuous at a, there is a $\delta > 0$ such that $f(B^{\rho}_{\delta}(a)) \subseteq B^{\tau}_{\epsilon}(f(a))$. But then, $B^{\rho}_{\delta}(a) \subseteq f^{-1}(f(B^{\rho}_{\delta}(a))) \subseteq f^{-1}(B^{\tau}_{\epsilon}(f(a)) \subseteq f^{-1}(\mathcal{O})$. So we have found an open ball about a that is contained in $f^{-1}(\mathcal{O})$.

Conversely, suppose that $f^{-1}(\mathcal{O})$ is open in A for every open subset $\mathcal{O}$ of B. We must show f is continuous. Let $a \in A$. To show f is continuous at a, let $\epsilon > 0$. We must find a $\delta > 0$ such that $f(B^{\rho}_{\delta}(a)) \subseteq B^{\tau}_{\epsilon}(f(a)$. But $B^{\tau}_{\epsilon}(f(a))$ is an open subset of B, and it follows by our assumption that $f^{-1}(B^{\tau}_{\epsilon}(f(a)))$ is open in A. Now, $a \in f^{-1}(B^{\tau}_{\epsilon}(f(a)))$, so there is a $\delta > 0$ such that $B^{\rho}_{\delta}(a) \subseteq f^{-1}(B^{\tau}_{\epsilon}(f(a)))$. But then $f(B^{\rho}_{\delta}(a)) \subseteq f(f^{-1}(B^{\tau}_{\epsilon}(f(a)))) \subseteq B^{\tau}_{\epsilon}(f(a))$. $\square$

This characterization of continuity gives another easy proof that the composition of continuous functions is continuous. The proof is left as an exercise. We finish with two theorems about compact sets. The first is a version of the Bolzano-Weirstrass theorem for sequences. The second says that the continuous image of a compact set is compact.

Theorem 4. Let (M, d) be a metric space, and suppose that K is a compact subset of M. Let $\{x_i\}_{i=1}^{\infty}$ be any sequence of points in K. Then $\{x_i\}_{i=1}^{\infty}$ has a subsequence that converges to a point in K.

Proof. If the terms of the sequence include only finitely many different points of K, then at least one of those points, say x, occurs infinitely often in the sequence. Then $\{x_i\}_{i=1}^{\infty}$ has a constant subsequence consisting of all the occurrences of x, and that subsequence converges to x, which is in K since every x_i is in K.

So, suppose that the terms of the sequence include infinitely many different points in K. Since K is compact, the infinite subset containing all of the terms of the sequence has an accumulation point in K (by Theorem 2 in Handout 2). Let z be that accumulation point. We show that $\{x_i\}_{i=1}^{\infty}$ has a subsequence that converges to z. We use the result from Exercise 5 in Handout 1, which implies that for any $\epsilon > 0$, the open ball $B_{\epsilon}(z)$ contains infinitely many of the terms of the sequence. Since z is an accumulation point of the set consisting of all the x_i, there is an index i_1 such that $x_{i_1} \in B_1(z)$. Then, since $B_{1/2}(z)$ contains infinitely many terms of the sequence, there is an index $i_2 > i_1$ such that $x_{i_2} \in B_{1/2}(z)$. Then, since $B_{1/3}(z)$ contains infinitely many terms of the sequence, there is an index $i_3 > i_2$ such that $x_{i_3} \in B_{1/3}(z)$. Proceeding in this way, we obtain indices $i_1 < i_2 < i_3 < \cdots$ such that $x_{i_n} \in B_{1/n}(z)$ for all $n \in \mathbb{N}$. It follows easily that the subsequence $\{x_{i_n}\}_{n=1}^{\infty}$ converges to z. □

Theorem 5. Let (A, ρ) and (B, τ) be metric spaces, and let $f\colon A \to B$ be a continuous function. Then for any compact subset K of A, the image $f(K)$ is a compact subset of B.

Proof. Let K be a compact subset of A. We show that any open cover of $f(K)$ has a finite subcover. Suppose that $\{\mathcal{O}_\alpha \,|\, \alpha \in \mathscr{A}\}$ is an open cover of $f(K)$ in B. By Theorem 3, each set $f^{-1}(\mathcal{O}_\alpha)$ is open in A. Furthermore, the collection $\{f^{-1}(\mathcal{O}_\alpha) \,|\, \alpha \in \mathscr{A}\}$ covers K (since for $k \in K$, $f(k) \in \mathcal{O}_\beta$ for some β, and then $k \in f^{-1}(\mathcal{O}_\beta)$). Since K is compact, it has a finite subcover, $\{f^{-1}(\mathcal{O}_{\alpha_1}), f^{-1}(\mathcal{O}_{\alpha_2}), \ldots, f^{-1}(\mathcal{O}_{\alpha_n})\}$. I claim that $\{\mathcal{O}_{\alpha_1}, \mathcal{O}_{\alpha_2}, \ldots, \mathcal{O}_{\alpha_n}\}$ is a finite subcover of $f(K)$. In fact, let $b \in f(K)$. Then $z = f(a)$ for some $a \in K$. Now, $a \in f^{-1}(\mathcal{O}_{\alpha_i})$ for some i. So, $b = f(a) \in f(f^{-1}(\mathcal{O}_{\alpha_i})) \subseteq \mathcal{O}_{\alpha_i}$. That is, every element of $f(K)$ is covered by some $\mathcal{O}_{\alpha_i}$. □

Exercises

Exercise 1. Let X be any set, and define the metric ρ on X as in Exercise 1 from Handout 1: $\rho\colon X \times X \to \mathbb{R}$ by $\rho(a,b) = \begin{cases} 0 & \text{if } a = b \\ 1 & \text{if } a \neq b \end{cases}$. Suppose that $\{x_i\}_{i=1}^{\infty}$ is a convergent sequence in the metric space (X, ρ), Show that there is a number N such that $x_N = x_{N+1} = x_{N+2} = \cdots$. (We say that the sequence is "eventually constant.")

Exercise 2. Let (X, ρ) be the metric space from the previous exercise, and let (M, d) be any metric space. Show that any function $f\colon X \to M$ is continuous. (There are at least three possible proofs: using the definition of continuity, using Theorem 1, or using Theorem 3.)

Exercise 3. Let $\{x_i\}_{n=1}^{\infty}$ be a convergent sequence in a metric space. Show that its limit is unique. That is, prove the following statement: if $\{x_i\}_{n=1}^{\infty}$ converges to y and $\{x_i\}_{n=1}^{\infty}$ converges to z, then $y = z$.

Exercise 4. Define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} x & \text{if } x \leq 1 \\ x+1 & \text{if } x > 1 \end{cases}$. Find a sequence $\{x_i\}_{i=1}^{\infty}$ that converges to 1, but $\{f(x_i)\}_{i=1}^{\infty}$ does not converge to $f(1)$. And find an open subset $\mathcal{O}$ of $\mathbb{R}$ such that $f^{-1}(\mathcal{O})$ is not open. ($\mathbb{R}$ here has its usual metric.)

Exercise 5. Let (M, d) be a metric space, and let $f\colon M \to \mathbb{R}$ and $g\colon M \to \mathbb{R}$ be two functions from M to $\mathbb{R}$ (where $\mathbb{R}$ has its usual metric). Let $x \in M$. Suppose f and g are continuous at x. Show that the function $f + g$ is continuous at x. (Hint: Just imitate the proof for functions from $\mathbb{R}$ to $\mathbb{R}$.)

Exercise 6. Use Theorem 3 to prove the composition of continuous functions between metric spaces is continuous.

Exercise 7. Let (M, d) be a non-empty compact metric space and let $f\colon M \to \mathbb{R}$ be a continuous function (where $\mathbb{R}$ has its usual metric). Show that $f(x)$ achieves a minimum value and a maximum value. This is a generalization of the Extreme Value Theorem. (Hint: Use the fact that $f(M)$ is compact, and apply Exercise 8 from Handout 1.)