Complete Metric Spaces

Math 331, Handout #5

We have looked at Cauchy sequences of real numbers, and we showed that every such sequence converges to a real number. Cauchy sequences can be defined in any metric space. It is not true, in general, that every Cauchy sequence in a metric space converges. A metric space in which that **is** true is said to be "complete." This handout looks at Cauchy sequences and completeness for general metric spaces. Many of the proofs, which can get somewhat complicated, will be omitted. A sequence is Cauchy if the terms of the sequence "get close to each other."

Definition 1. Let (M, d) be a metric space. A sequence, $\{x_i\}_{i=1}^{\infty}$, of elements of M is said to be a **Cauchy sequence** if and only if for any $\epsilon > 0$, there is a number N such that if n and m are integers greater than or equal to N, then $d(x_n, x_m) < \epsilon$.

Theorem 1. Any convergent sequence in any metric space is a Cauchy sequence.

Proof. The proof for general metric spaces is essentially the same as the proof for $\mathbb{R}$ and is left as an exercise. □

Definition 2. A metric space (M, d) is said to be **complete** if and only if every Cauchy sequence of elements of M converges to an element of M.

We have already seen that $\mathbb{R}$ is complete in this sense, as well as the original sense in which we used the term, that is that every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound. For a general metric space, of course, we can't talk about least upper bounds, since there is no concept of "less than" in a typical metric space.

A closed subset of a complete metric space is itself complete, when considered as a subspace using the same metric, and conversely. We state this without proof. Note that this means that a closed, bounded interval in $\mathbb{R}$ is a complete metric space. Similarly, the Cantor set is complete.

Theorem 2. Let (M, d) be a complete metric space, and let X be a subset of M. The subspace (X, d) is a complete metric space if and only if X is a closed subset of M.

* * *

We now turn to the relationship between completeness and compactness. These two concepts are not the same; for example, $\mathbb{R}$ is complete but not compact. However, it is true that any compact metric space is complete.

Theorem 3. Let (K, d) be a compact metric space. Then (K, d) is a complete metric space.

Proof. Suppose (K, d) is compact. Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in K. We show that $\{x_n\}_{n=1}^{\infty}$ converges, thus proving K is complete. By Theorem 4 in Handout 3, the sequence has a subsequence that converges to some element z in K. By Exercise 2 in this handout, the sequence itself must converge to z. □

We can get a full characterization of compactness by adding an additional property to completeness. Exercise 2 in Handout 2 showed that every compact metric space is "totally bounded." In fact, a metric space is compact if and only if it is both complete and totally bounded. We start with the formal definition of totally bounded, followed by a statement of the theorem that compactness is equivalent to being complete and totally bounded.

Definition 3. A metric space (M, d) is said to be totally bounded if for every $\epsilon > 0$, there is a finite subset $\{x_1, x_2, \ldots, x_n\}$ of M (for some n) such that $\{B_\epsilon(x_1), B_\epsilon(x_2), \ldots, B_\epsilon(x_n)\}$ is an open cover of M.

Theorem 4. A metric space is compact if and only if it is both complete and totally bounded.

As noted in handout 2, every closed, bounded subset of $\mathbb{R}$ is compact. We can now see that this is true because every closed subset of $\mathbb{R}$ is complete, by Theorem 2, and every bounded subset of $\mathbb{R}$ is totally bounded by the following theorem:

Theorem 5. Every bounded subset of $\mathbb{R}$ is a totally bounded.

Proof. Let X be a bounded subset of $\mathbb{R}$. Let $\epsilon > 0$. Then $X \subseteq [a, b]$ for some $a, b \in \mathbb{R}$. The interval $[a, b]$ can be covered with a finite number of open subintervals of length ϵ, and those subintervals will also cover X. □

Accepting our new characterization of compactness as true, we are in a position to prove a claim that was made at the end of Handout 2: A metric space is compact if and only if every infinite subset has an accumulation point. We can also add another equivalent condition that uses sequences, as well as the condition from Theorem 5:

Theorem 6. Let (M, d) be a metric space. The following statements are equivalent:

(1) M is compact. (That is, every open cover of M has a finite subcover.)
(2) Every infinite subset of M has an accumulation point.
(3) Every infinite sequence in M has a convergent subsequence.
(4) M is complete and totally bounded.

Proof. The implication (1) $\Rightarrow$ (2) was proved in Theorem 2 of Handout 2, while (1) $\Rightarrow$ (3) was proved in Theorem 4 of Handout 3, and (1) $\iff$ (4) is Theorem 5.

To prove (3) $\Rightarrow$ (4), suppose that every infinite sequence in M has a convergent subsequence. To show M is complete, let $\{x_i\}_{i=1}^\infty$ be a Cauchy sequence in M. By the assumption, it has a convergent subsequence. By Exercise 2, $\{x_i\}_{i=1}^\infty$ is also convergent, to the same limit. So, M is complete. We use proof by contradiction to show that M is totally bounded. Suppose that M is **not** totally bounded. Then there is an $\epsilon > 0$ such that M cannot be covered by a finite number of open balls of radius ϵ. Let $x_1 \in M$. Since M is not covered by $B_\epsilon(x_1)$, there is an $x_2 \in M$ such that $d(x_2, x_1) \geq \epsilon$. Since $B_\epsilon(x_1)$ and $B_\epsilon(x_2)$ do not cover M, there is an $x_3 \in M$ such that $d(x_3, x_1) \geq \epsilon$ and $d(x_3, x_2) \geq \epsilon$. Continuing in this way, we get a sequence $\{x_n\}_{n=1}^\infty$ such that the distance between any two terms in the sequence is greater than or equal to ϵ. But no subsequence of such a sequence can be convergent, contradicting the hypothesis.

To complete the proof, we can show that (2) $\Rightarrow$ (3). But that left as Exercise 3. □

We finish by looking at what can be done with Cauchy sequences in metric spaces that are not necessarily complete. Let (M, d) be any metric space. Start with a set $\mathscr{C}(M)$ whose elements are all the Cauchy sequences in M. Define a relation, $\sim$, on $\mathscr{C}(M)$ by $\{a_i\}_{i=1}^\infty \sim \{b_i\}_{i=1}^\infty$ if and only if $\lim_{i\to\infty} d(a_i, b_i) = 0$. This relation is an equivalence relation on $\mathscr{C}(M)$. (The proof is left as an exercise.) Now, consider the set $\mathscr{C}[M]$ of equivalence classes in $\mathscr{C}(M)$ under that relation. We can define a metric ∂ on $\mathscr{C}[M]$ by $\partial\big([\{x_i\}_{i=1}^{\infty}], [\{y_i\}_{i=1}^{\infty}]\big) = \lim_{i\to\infty} d(x_i, y_i)$. (The long proof that ∂ is well-defined and is a metric is omitted.) This gives us a new metric space, $(\mathscr{C}[M], \partial)$.

We can define a function $f\colon M \to \mathscr{C}[M]$ by letting $f(x)$ be the equivalence class of the constant sequence in which every term is x; that is, $f(x) = \big[\{x\}_{i=1}^{\infty}\big]$. This function has the property that $\partial(f(x), f(y)) = d(x, y)$. It is clearly continuous and one-to-one, and it allows us to identify the metric space (M, d) with the subspace $f(M)$ of $(\mathscr{C}[M], \partial)$.

Things get even more complicated after that, so I won't go into any further details here. However, it can be shown that $(\mathscr{C}[M], \partial)$ is a complete metrics space and that $f(M)$ is dense in $\mathscr{C}[M]$. And if the metric space (M, d) is already complete, then $(\mathscr{C}[M], \partial)$ is essentially just a copy of M, since in that case the function f is an isometry—a bijection that preserves distance. The space $(\mathscr{C}[M], \partial)$ is said to be the **completion** of (M, d).

If we start with the rational numbers, $\mathbb{Q}$, with their usual metric, it turns out that $\mathscr{C}[\mathbb{Q}]$ is isometric with $\mathbb{R}$. This means that it is possible to define $\mathbb{R}$ as a set of equivalence classes of Cauchy sequences of rational numbers. We started this course with a construction of $\mathbb{R}$ as a set of Dedekind cuts of $\mathbb{Q}$. Cauchy sequences provide an alternative construction, but the structure defined in that way has identical properties to the structure defined by Dedekind cuts.

Exercises

Exercise 1. Prove Theorem 1.

Exercise 2. Suppose that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in some metric space (M, d), and suppose that the sequence has a subsequence that converges to $z \in M$. Show that $\{x_n\}_{n=1}^{\infty}$ converges to z.

Exercise 3. Complete the proof of Theorem 6 by proving that if every infinite subset of a metric space has an accumulation point, then every infinite sequence in that metric space has a convergent subsequence. (Hint: Consider two cases: the sequence contains only finitely many different terms, and the sequence contains an infinite number of different terms.)

Exercise 4. Let (M, d) be metric space. Show that the relation $\sim$ on $\mathscr{C}[M]$ is an equivalence relation.

Exercise 5. Let (M, d) be a metric space. A function $f\colon M \to M$ is said to be a **contraction** if there is a number r with $0 \le r < 1$ such that for any $x \in M$ and $y \in M$, $d(f(x), f(y)) \le r \cdot d(x, y)$.

(a) Show that any contraction is continuous.

(b) Let f be a contraction on a metric space (M, d), and let $x \in M$. Show that the sequence $\{x, f(x), f(f(x)), f(f(f(x))), \dots, f^{\circ n}(x), \dots\}$ is Cauchy, where $f^{\circ n}$ is the composition of f with itself n times. (Hint: If $0 < r < 1$, and $n \ge m$, then $\sum_{i=m}^{n} r^i = r^m \cdot \sum_{k=0}^{n-m} r^k < r^m \cdot \sum_{k=0}^{\infty} r^k = r^m \cdot \frac{1}{1-r}$.)

(c) Suppose (M, d) is a complete metric space and $f\colon M \to M$ is a contraction. Let $x \in M$. Part (b) implies that $\lim_{n\to\infty} f^{\circ n}(x)$ exists. Let $z = \lim_{n\to\infty} f^{\circ n}(x)$. Show z is a fixed point of f, that is, $f(z) = z$.

(d) Suppose (M, d) is a non-empty complete metric space and $f\colon M \to M$ is a contraction. Show that f has a **unique** fixed point and that for every $x \in M$, the sequence $\{f^{\circ n}(x)\}_{n=0}^{\infty}$ converges to that fixed point. This is the **Contraction Theorem** for complete metric spaces.

Exercise 6. Show that ∂ is a well-defined function on $\mathscr{C}[M] \times \mathscr{C}[M]$. That is, show that if $\{a_i\}_{i=1}^{\infty} \sim \{b_i\}_{i=1}^{\infty}$ and $\{x_i\}_{i=1}^{\infty} \sim \{y_i\}_{i=1}^{\infty}$, then $\partial\big([\{a_i\}_{i=1}^{\infty}], [\{x_i\}_{i=1}^{\infty}]\big) = \partial\big([\{b_i\}_{i=1}^{\infty}], [\{y_i\}_{i=1}^{\infty}]\big)$. Then show ∂ is a metric. [This is a long an non-trivial exercise]