Math 331, Homework 10

This homework covers Handout #5 and Sections 4.3 and 4.4 and is due Friday, November 22.

Exercise 1 (Exercise 4.3.5 in the textbook)**.** Prove: If $\sum_{k=1}^{\infty} a_k$ converges and $\sum_{k=1}^{\infty} b_k$ diverges, then $\sum_{k=1}^{\infty}(a_k + b_k)$ diverges.

Exercise 2 (Exercise 4.3.8 in the textbook)**.** Show: If $\sum_{k=1}^{\infty} a_k$ is a convergent series, there there exists an integer N such that if $n > N$, then $\left|\sum_{k=n+1}^{\infty} a_k\right| < \epsilon$. That is, the infinite tail of the series can be made arbitrarily small. (Hint: Use the Cauchy criterion for convergence, from Page 182.)

Exercise 3 (mostly from Exercises 4.3.3)**.** Determine whether the following series converge or diverge. For alternating series that converge, determine whether the series is conditionally or absolutely convergent. Be explicit about which tests are applied and how.

a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k + |\sin k|}$ **b)** $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{5^k}$ **c)** $\displaystyle\sum_{k=1}^{\infty} \frac{k+1}{3k+1}$ **d)** $\displaystyle\sum_{k=1}^{\infty} \frac{k^2-3}{2+k^5}$

e) $\displaystyle\sum_{k=1}^{\infty} \pi^{-k}$ **f)** $\displaystyle\sum_{k=1}^{\infty} \frac{k^5}{2^k}$ **g)** $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^5+1}}$ **h)** $\displaystyle\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}$

i) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{2k-1}$ **j)** $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln(k)}$ **k)** $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{2^k}$ **l)** $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k k!}{k^k}$

Exercise 4 (Handout #5, Exercise 5)**.** Let (M, d) be a metric space. A function $f\colon M \to M$ is said to be a **contraction** if there is a number r with $0 \le r < 1$ such that for any $x \in M$ and $y \in M$, $d(f(x), f(y)) \le r \cdot d(x, y)$.

(a) Show that any contraction is continuous.

(b) Let f be a contraction on a metric space (M, d), and let $x \in M$. Show that the sequence $\{x, f(x), f(f(x)), f(f(f(x))), \ldots, f^{\circ n}(x), \ldots\}$ is Cauchy, where $f^{\circ n}$ is the composition of f with itself n times. (Hint: See the proof of Corollary 4.2.7, the Contraction Principle for Sequences, in the textbook. The proof uses the following fact: If $0 < r < 1$, and $n \ge m$, then $\sum_{i=m}^{n} r^i = r^m \cdot \sum_{k=0}^{n-m} r^k < r^m \cdot \sum_{k=0}^{\infty} r^k = r^m \cdot \frac{1}{1-r}$.)

(c) Suppose (M, d) is a complete metric space and $f\colon M \to M$ is a contraction. Let $x \in M$. Part (b) implies that $\lim\limits_{n\to\infty} f^{\circ n}(x)$ exists. Let $z = \lim\limits_{n\to\infty} f^{\circ n}(x)$. Show z is a fixed point of f, that is, $f(z) = z$.

(d) Suppose (M, d) is a non-empty complete metric space and $f\colon M \to M$ is a contraction. Show that f has a **unique** fixed point and that for every $x \in M$, the sequence $\{f^{\circ n}(x)\}_{n=0}^{\infty}$ converges to that fixed point. This is the **Contraction Theorem** for complete metric spaces.

Exercise 5 (Handout #5, Exercise 2)**.** Suppose that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in some metric space (M, d), and suppose that the sequence has a subsequence that converges to $z \in M$. Show that $\{x_n\}_{n=1}^{\infty}$ converges to z.

Exercise 6 (from Handout #5, Exercise 4)**.** Let (M, d) be metric space. We defined $\mathscr{C}(M)$ to be the set of Cauchy sequences in M, and for two Cauchy sequences, $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$, we defined $\{x_n\}_{n=1}^{\infty} \sim \{y_n\}_{n=1}^{\infty}$ if and only if $\lim_{n\to\infty} d(x_n, y_n) = 0$. Show that $\sim$ is an equivalence relation, that is that it is reflexive, symmetric, and transitive.

Exercise 7 (For Extra Credit, based on Handout #5)**.**

(a) Let (M, d) be a metric space, and suppose that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are Cauchy sequences in M. Show that the sequence $\{d(x_n, y_n)\}_{n=1}^{\infty}$ is a Cauchy sequence of real numbers, and hence converges to some real number. [Hint: It is possible to show that for any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $m, n > N$, $d(x_n, y_n) < d(x_m, y_m) + \epsilon$. Draw a picture!]

(b) Let (M, d) be a metric space. We defined $\mathscr{C}[M]$ as the set of equivalence classes of $\mathscr{C}(M)$ under the $\sim$ relation. And we defined a metric ∂ on $\mathscr{C}[M]$ by $\partial\big([\{x_i\}_{i=1}^{\infty}], [\{y_i\}_{i=1}^{\infty}]\big) = \lim_{n\to\infty} d(x_n, y_n)$. (Part (a) shows that this limit exists.) Show that ∂ is a well-defined function on $\mathscr{C}[M] \times \mathscr{C}[M]$. That is, show that if $\{a_i\}_{i=1}^{\infty} \sim \{b_i\}_{i=1}^{\infty}$ and $\{x_i\}_{i=1}^{\infty} \sim \{y_i\}_{i=1}^{\infty}$, then $\partial\big([\{a_i\}_{i=1}^{\infty}], [\{b_i\}_{i=1}^{\infty}]\big) = \partial\big([\{x_i\}_{i=1}^{\infty}], [\{y_i\}_{i=1}^{\infty}]\big)$.

(c) Show ∂ is a metric.