Math 331, Homework 6

This homework covers the sequences and continuity handout and Section 2.6. It is due next Friday, October 11.

Exercise 1. Let X be any set, and define the discrete metric ρ on X as in Exercise 1 from Handout 1: $\rho\colon X \times X \to \mathbb{R}$ by $\rho(a,b) = \begin{cases} 0 & \text{if } a = b \\ 1 & \text{if } a \neq b \end{cases}$. Suppose that $\{x_i\}_{i=1}^{\infty}$ is a convergent sequence in the metric space (X,ρ), Show that there is a number N such that $x_N = x_{N+1} = x_{N+2} = \cdots$. (We say that the sequence is "eventually constant.")

Exercise 2. Let (X,ρ) be the metric space from the previous exercise, and let (M,d) be any metric space. Show that any function $f\colon X \to M$ is continuous. (There are at least three possible proofs: using the definition of continuity, using Theorem 1 from the handout, or using Theorem 3 from the handout.)

Exercise 3. Define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} x & \text{if } x \le 1 \\ x+1 & \text{if } x > 1 \end{cases}$. Find a sequence $\{x_i\}_{i=1}^{\infty}$ that converges to 1, but $\{f(x_i)\}_{i=1}^{\infty}$ does not converge to $f(1)$. And find an open subset $\mathcal{O}$ of $\mathbb{R}$ such that $f^{-1}(\mathcal{O})$ is not open. ($\mathbb{R}$ here has its usual metric.)

Exercise 4. Let (M,d) be a metric space, and let $f\colon M \to \mathbb{R}$ and $g\colon M \to \mathbb{R}$ be two functions from M to $\mathbb{R}$ (where $\mathbb{R}$ has its usual metric). Let $x \in M$. Suppose f and g are continuous at x. Show that the function $f+g$ is continuous at x. (Hint: Just imitate the proof for functions from $\mathbb{R}$ to $\mathbb{R}$.)

Exercise 5. Let (M,d) be a non-empty compact metric space and let $f\colon M \to \mathbb{R}$ be a continuous function (where $\mathbb{R}$ has its usual metric). Show that $f(x)$ achieves a minimum value and a maximum value. This is a generalization of the Extreme Value Theorem, which our textbook calls the Min-Max Theorem. (Hint: Use the fact that $f(M)$ is compact—and therefore closed and bounded—and apply Exercise 8 from Handout 1.)

Exercise 6 (Problem 2.6.7b from the textbook)**.** Show that the function $p(x) = x^4 - x^3 + x^2 + x - 1$ has at least two roots in the interval $[-1,1]$.

Exercise 7 (Problem 2.6.2 from the textbook)**.** Show that a linear function $f(x) = mx + b$ is uniformly continuous on $(-\infty,\infty)$.

Exercise 8 (Problem 2.6.10ac from the textbook)**.**

(a) Prove: If f is uniformly continuous on the bounded open interval (a,b), then f is bounded on (a,b).

(b) Why must the interval in part (a) be bounded?

Exercise 9 (Problem 2.6.12ab from the textbook)**.** We say that a function f satisfies a "Lipschitz condition" if there is a positive real number M such that for all $x, y \in \mathbb{R}$, $|f(x) - f(y)| < M \cdot |x-y|$.

(a) Show that any function that satisfies a Lipschitz condition is uniformly continuous on $(-\infty,\infty)$.

(b) Show that the function $f(x) = \dfrac{1}{1+|x|}$ is uniformly continuous on $(-\infty,\infty)$.