Math 331, Homework 9

This homework covers mainly Sections 3.7 and 4.2 and is due Friday, November 15.

Exercise 1. In class, we used the generalized mean value theorem for integrals to derive the Lagrange form of the remainder for Taylor's Theorem from the integral form. Prove the generalized mean value theorem for integrals: Let f and g be continuous function on $[a, b]$, and suppose that $g(x) \geq 0$ for all $x \in [a, b]$. Then there is a $c \in (a, b)$ such that $\int_a^b fg = f(c) \int_a^b g$. (This is Exercise 3.6.10, parts (a) and (b), in the textbook, and there are some hints there.)

Exercise 2. Find the general Taylor polynomial at 0, $p_{n,0}(x)$, for the function $\ln(x + 1)$.

Exercise 3. Let f be a function that has a continuous 4^{th} derivative near $x = 1$, and suppose that its Taylor polynomial of degree 3 at 1 is $p_{3,1}(x) = 4 - 2x + 3x^2 + 5x^3$. Define the function $g(x) = f(x^2)$. Find the numeric value of $g'''(1)$.

Exercise 4. Let f be the polynomial $f(x) = x^4 - 2x^3 + x^2 - 1$. Use Taylor's Theorem to write f as a polynomial in powers of $(x+1)$. (That is, find the Taylor polynomial of degree 4 at -1 for f.)

Exercise 5 (4.2.2 from the textbook)**.** For each of the following cases, find a sequence that satisfies the given restrictions. If such a combination is imposible, briefly state why.

(a) A sequence that is monotone but not convergent.

(b) A sequence that is bounded but not monotone.

(c) A sequence that is monotone but not Cauchy.

(d) A sequence that is monotone and bounded but not Cauchy.

Exercise 6 (4.2.3 from the textbook)**.** Prove that $\{p_n\}_{n=1}^{\infty}$ converges where $p_n = \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$.

Exercise 7 (from 4.2.13 from the textbook)**.** For this problem, assume that we know that the function $f(x) = b^x$ is continuous, and that it is increasing if $b > 1$.

Let b be a positive real number and define the sequence $\{a_n\}_{n=1}^{\infty}$ by

$$a_1 = b, \qquad a_2 = b^b, \qquad a_3 = b^{b^b}, \ldots, \qquad a_n = b^{a_{n-1}}, \ldots$$

(a) Show that if $b > 1$, then $\{a_n\}_{n=1}^{\infty}$ is increasing. (Hint: If $a_{k+1} > a_k$ and $b > 1$, then $b^{a_{k+1}} > b^{a_k}$. Use induction.)

(b) Show that if $1 < b \leq e^{1/e}$, then $\{a_n\}_{n=1}^{\infty}$ converges to some number a. (Hint: Show by induction that $a_n \leq e$ for all n.) Further, show that $a = b^a$. (Hint: Use $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1}$.)

(c) Show that $2 = \sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot}}}}}$

Exercise 8 (from 4.2.14 from the textbook)**.** We have shown that the n^{th} Taylor polynomial for e^x at 0 is $p_{n,0}(x) = \sum_{k=1}^{n} \frac{1}{n!} x^n$. Show that e is irrational by using proof by contradiction. Suppose, for the sake of contradiction, that $e = \frac{p}{q}$ for some integers p and q.

(a) Use the remainder term from Taylor's Theorem to show that there is a $c \in [0, 1]$ such that $\frac{p}{q} - \left(\frac{1}{0!} + \frac{1}{1!} + \cdots + \frac{1}{n!}\right) = \frac{e^c}{(n+1)!}$.

(b) Multiply both sides of the equation in (a) by $n!$, and show that left side of the resulting equation is an integer when $n \geq q$.

(c) Show that the right side of the equation that you got in part (b) is not an integer when $n > e$. Conclude that e is irrational.