Math 331, Fall 2019, Some suggested problems on integration

These problems are not homework; solutions will not be collected.
*These problems are **not** meant as a review for the test.*

1. **(a)** Let $c \in [a,b]$, let $y \in \mathbb{R} \setminus \{0\}$, and define a function p on $[a,b]$ by $p(c) = y$ and $p(x) = 0$ for $x \neq c$. Show that f is integrable on $[a,b]$ and that $\int_a^b f = 0$.

(b) Let h be a function on $[a,b]$ such that $h(x) = 0$ except at finitely many points. Show h is integrable and $\int_a^b h = 0$. (You can use part (a), induction, and the linearity of integration.)

(c) Suppose f and g are functions on $[a,b]$. Suppose that f is integrable and $g(x) = f(x)$ except at finitely many points in $[a,b]$. Show that g is integrable on $[a,b]$ and $\int_a^b f = \int_a^b g$. (Hint: Consider $g - f$.)

2. Prove carefully using the defintion of Riemann integral that if f is a continuous function on $[a,b]$, and $f(x) \geq 0$ for all $x \in [a,b]$, and there is a $c \in [a,b]$ for which $f(c) > 0$, then $\int_a^b f > 0$. Does the conclusion still hold if f is only assumed to be integrable rather than continuous?

3. We have used the following fact; you should prove it carefully, using the definitions of inf, sup, and the Riemann integral: Let f be an integrable function on $[a,b]$, and suppose that I is a number satisfying $L(P;f) \leq I \leq U(P;f)$ for every partition P of $[a,b]$. Then $I = \int_a^b f$.

4. Suppose f is integrable on $[a,b]$. Prove that $|f|$ is integrable on $[a,b]$. Here is an outline of a proof: Let $\epsilon > 0$. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$ such that $U(P;f) - L(P;f) < \epsilon$. As usual, let $M_i = \sup\{f(x) \,|\, x \in [x_{i-1}, x_i]\}$, and let $m_i = \inf\{f(x) \,|\, x \in [x_{i-1}, x_i]\}$, so that $U(P;f) - L(P;f) = \sum_{i=0}^n (M_i - m_i)(x_i - x_{i-1}) < \epsilon$. Let $M_i' = \sup\{|f(x)| \,|\, x \in [x_{i-1}, x_i]\}$, and let $m_i' = \inf\{|f(x)| \,|\, x \in [x_{i-1}, x_i]\}$. We then have $U(P;|f|) - L(P;|f|) = \sum_{i=0}^n (M_i' - m_i')(x_i - x_{i-1})$. Show that $M_i' - m_i' \leq M_i - m_i$, using properties of inf and sup and the fact that $\big||f(x)| - |f(y)|\big| \leq |f(x) - f(y)|$. And use that to deduce that $U(P;|f|) - L(P;|f|) < \epsilon$.

5. Suppose that f is a function on $[a,b]$ and that $|f|$ is integrable on $[a,b]$. Is f necessarily integrable on $[a,b]$?

6. Suppose that f is integrable on $[a,b]$. By problem 4, above, $|f|$ is integrable on $[a,b]$. Use problem 3.4.8 and the fact that $-f(x) \leq |f(x)| \leq f(x)$ to prove that $\left|\int_a^b f\right| \leq \int_a^b |f|$.

7. Suppose that f is integrable on $[a,b]$. Define $F(x) = \int_a^x f$ for $x \in [a,b]$, and define $G(x) = \int_a^x F$ for $x \in [a,b]$. (How do we know $\int_a^x F$ exists?) Show that G is differentiable on $[a,b]$.

8. Let g be a bounded function on $[a,b]$. Let $P = \{x_o, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. Suppose that $U(P;g) = L(P;g)$. Explain why this means g is integrable. What is $\int_a^b g$? What does g actually look like?