This test is due in class on Wednesday, October 23. You should do this test on your own, using only your textbook, class notes, and previous work in the course as reference. You should not work with other students, and you should not consult anyone except me. You can ask questions about the test in class, in office hours, and by email. I might give some hints and clarifications, but I am unlikely to give extensive, detailed help.

This test counts for 15% of your overall grade for the course. There are 8 problems. Although some problems are more difficult than others, all problems will count equally. Note that for the question number 8, you have a choice of doing either 8-A or 8-B.

Note: There will be no "rewrites" of midterm exam problems. *Be sure to show all of your work!*

1. In class, we proved directly that for $n \in \mathbb{N}$, $\frac{d}{dx}x^n = nx^{n-1}$, using the definition of derivative. Give an alternative proof of this fact using mathematical induction, the product rule for derivatives, and the fact that $\frac{d}{dx}x = 1$.

2. In class, we proved the product rule for derivatives directly from the definition of derivative. This problem asks you to provide a different proof. You should assume that the sum and constant multiple rules for derivatives have already been proved.

a) Let f be a function that is differentiable at a, and let $g(x) = f(x)^2$. Show directly, using the definition of derivative, that g is differentiable at a and that $g'(x) = 2f(x)f'(x)$.

b) Recall that $ab = \frac{1}{4}\big((a+b)^2 - (a-b)^2\big)$. Use this fact and part (a) of this problem to prove the product rule for derivatives.

3. Suppose that $f(x)$ and $g(x)$ are uniformly continuous on the interval I (which is not necessarily closed or bounded). Show directly from the definition of uniform continuity that $f(x) + g(x)$ is uniformly continuous on I.

4. **a)** Show directly from the definition of convergence of an infinite sequence that $\lim\limits_{n\to\infty} \frac{1}{n} = 0$.

b) Use the result from part (a) and Theorem 4.1.5 (properties of limits of sequences) to find the following limit. (Start by multiplying numerator and denominator by $\frac{1}{n^4}$).

$$\lim_{n\to\infty} \frac{3n^4 - 2n + 7}{5n^4 + 3n^3 - 1}$$

5. Let A and B be subsets of $\mathbb{R}$. Suppose that x is an accumulation point of the set $A \cup B$. Show that x is an accumulation point of A or that x is an accumulation point of B (or both). (Hint: Try a proof by contradiction.) If you prefer to work with subsets of a metric space (M, d) instead of $\mathbb{R}$, you can do that instead.

6. Let X be a bounded, non-empty subset of $\mathbb{R}$. Suppose that X has the following property: If $a \in X$ and $b \in X$ and c is some number satisfying $a < c < b$, then $c \in X$. Prove that X must be a bounded interval of one of the forms $[x, y]$, $[x, y)$, $(x, y]$, or (x, y). (Hint: This is harder than it looks because a set does not necessarily contain its least upper bound or greatest lower bound.)

7. Let (M, d) be a metric space, and let X be a subset of M. This question explores several alternative definitions for what it means for X to be *dense* in M. Prove that the following are equivalent:

a) For every open ball $B_\epsilon(x)$ in M, the intersection $X \cap B_\epsilon(x)$ is not empty.

b) For every non-empty open set $U \subseteq M$, the intersection $X \cap U$ is not empty.

c) $\overline{X} = M$ (where $\overline{X}$ is the closure of X).

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8. As the last problem of the test, you should do **one** of the following two alternatives, 8-A or 8-B. Each alternative proves an interesting result! (If you would like to work on both alternatives, you can get a small amount of extra credit by doing both correctly.)

8-A. This question uses the Heine-Borel theorem to prove a somewhat surprising property of open covers of a closed interval $[a, b]$. Given an open cover of $[a, b]$, not only is every point $z \in [a, b]$ in one of the open sets in the cover, but in one of the open sets that contain z, z is not too close to the boundary of the open set. Stated formally, we have the following theorem:

Theorem. Suppose $\{\mathcal{O}_\alpha \,|\, \alpha \in A\}$ is an open cover of the closed, bounded interval $[a, b]$. Then there is a $\lambda > 0$ such that for every $z \in [a, b]$, there is an open set $\mathcal{O}_\alpha$ in the open cover that contains the entire interval $(z - \lambda, z + \lambda)$.

Fill in the details in the following proof of this theorem. Note that only parts (a) and (c) are non-trivial, and even those two parts are not hard—but you do have to pay careful attention to where a factor of 2 is included and where it is not. (As an alternative to working in $\mathbb{R}$, you can state and prove the corresponding theorem for compact subsets of metric spaces. The only difference is that open intervals $(x - \epsilon, x + \epsilon)$ are replaced by open balls $B_\epsilon(x)$. In fact, the notation for metric spaces is probably a little easier to handle!)

a) For each $x \in [a, b]$, there is a $\lambda_x > 0$ such that $(x - 2\lambda_x, x + 2\lambda_x) \subseteq \mathcal{O}_\alpha$ for some α.

b) The intervals $I_x = (x - \lambda_x, x + \lambda_x)$, for $x \in [a, b]$, cover $[a, b]$ and so there is a finite subcover $I_{x_1}, I_{x_2}, \dots, I_{x_n}$. Let $\lambda = \min(\lambda_{x_1}, \lambda_{x_2}, \dots, \lambda_{x_n},)$.

c) Let $z \in [a, b]$, and choose i such that $z \in (x_i - \lambda_{x_i}, x_i + \lambda_{x_i})$. Then $(z - \lambda, z + \lambda) \subseteq (x_i - 2\lambda_{x_i}, x_i + 2\lambda_{x_i})$.

d) $(z - \lambda, z + \lambda) \subseteq \mathcal{O}_\alpha$ for some α.

8-B. This question concerns infinite sequences. You can write the proof for sequences in $\mathbb{R}$ or for sequences in a metric space. The proof would be essentially the same in either case; only the notation changes slightly.

Let $\{x_n\}_{n=1}^{\infty}$ be an infinite sequence. We define a *rearrangement* of the sequence as follows: Let $s\colon \mathbb{N} \to \mathbb{N}$ be a bijective function. Then $\{x_{s(i)}\}_{i=1}^{\infty}$ is a rearrangement of the sequence $\{x_n\}_{n=1}^{\infty}$. The rearranged sequence has exactly the same terms as the original sequence, just in a different order.

Suppose that $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence and that $\lim\limits_{n\to\infty} x_n = L$, and let $\{x_{s(i)}\}_{i=1}^{\infty}$ be a rearrangement of the sequence. Show that the rearranged sequence $\{x_{s(i)}\}_{i=1}^{\infty}$ is convergent and converges to the same limit, $\lim\limits_{i\to\infty} x_{s(i)} = L$. (Hint: This is easier than it looks. Note that the set $\{x_1, x_2, \dots, x_N\}$ is finite.)