Math 331, Fall 2019, Test 1 Information

The first in-class test will take place on Wednesday, September 25. It will cover the following material: from the beginning of the textbook through Section 2.2; Section 4.1; and the handouts on metric spaces and compactness. You can expect questions that cover concepts, definitions, and theorems. You should certainly be able to give clear and exact statements of important definitions and theorems. You should be able to work with the concepts, definitions, and theorems as they apply to examples and problems. You should be able to do proofs that are, from my point of view at least, simple and straightforward. The questions on the test will include shorter and longer essay-type questions, math problems (things like "find the least upper bound" or "find the following limit"), and simple proofs. There might be some true/false or "prove or disprove" questions. And maybe I will come up with something more clever.

Some things that you should know about for the test:

the sets $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{N}$

rational and irrational numbers

examples of irrational numbers, such as $\sqrt[k]{p}$ for $k \geq 2$ and p prime

Dedekind cuts; defining $\mathbb{R}$ as the set of Dedekind cuts of $\mathbb{Q}$

bounded sets in $\mathbb{R}$; upper bounds and lower bounds

least upper bounds and greatest lower bounds

how to find the least upper bound of a bounded-above, non-empty set of Dedekind cuts

Archimedean property of $\mathbb{R}$

$\mathbb{R}$ is a complete, ordered field (and is characterized by this property)

a field is set with multiplication and addition satisfying certain axioms

an ordered field has an operation $<$ that is defined by a set, P, of positive elements

absolute value, $|x|$, and distance in $\mathbb{R}$, $|x-y|$

triangle inequality in $\mathbb{R}$: $|a+b| < |a| + |b|$, or $|x-z| < |x-y| + |y-z|$

open covers and subcovers

accumulation point of a set in $\mathbb{R}$ or in a metric space

infinite sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers; convergent sequence; limit of a sequence

$\lim\limits_{x \to a} f(x)$

metric space; distance in a metric space; open ball $B_\epsilon(x)$

open subset in a metric space; closed subset in a metric space

closure of a subset of a metric space

compact subset of a metric space; compact metric space

bounded set in a metric space; diameter of a set in a metric space

Definition. A **Dedekind cut** is a subset, α, of $\mathbb{Q}$ satisfying (1) $\alpha \neq \varnothing$ and $\alpha \neq \mathbb{Q}$; (2) if $p \in \alpha$ and $q < p$, then $q \in \alpha$; and (3) if $p \in \alpha$, then there is some $r \in \alpha$ such that $r > p$.

Definition. A subset S of $\mathbb{R}$ is **dense** in $\mathbb{R}$ if for any $a, b \in \mathbb{R}$ with $a < b$, there exists an $s \in S$ such that $a < s < b$. (Alternative definition: S is dense if for every non-empty open subset U of $\mathbb{R}$, $S \cap U \neq \varnothing$.)

Definition. An **accumulation point** of a subset X of a metric space is a point a such that for any $\epsilon > 0$, $X \cap (B_\epsilon(a) \smallsetminus \{a\}) \neq \varnothing$. (Equivalently, for a subset X of $\mathbb{R}$, a is an accumulation point of X if for any $\epsilon > 0$, there is an $x \in X$ such that $0 < |x - a| < \epsilon$.)

Definition. A sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers **converges to** L if for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $|x_n - L| < \epsilon$ for all $n \geq N$.

Definition. If $f\colon \mathbb{R} \to \mathbb{R}$ and $a, L \in \mathbb{R}$, we say $\lim\limits_{x \to a} f(x) = L$ if for every $\epsilon > 0$, there is a $\delta > 0$ such that for any x, $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$.

Definition. A **metric space** is a pair (M, d) where M is a set and $d\colon M \times M \to \mathbb{R}$ satisfying (1) $d(x, x) = 0$ for all $x \in M$ and $d(x, y) > 0$ for all $x \neq y$ in M; (2) $d(x, y) = d(y, x)$ for all $x, y \in M$; and (3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in M$.

Definition. A subset U of a metric space (M, d) is **open** if for every $x \in U$, there is an $\epsilon > 0$ such that $B_\epsilon^d(x) \subseteq U$. A subset C is **closed** if $M \smallsetminus C$ is open.

Definition. A subset K of a metric space (M, d) is **compact** if every open cover of K (by open subsets of M) has a finite subcover. If M itself is compact, we say that M is a **compact metric space**.

Theorem. (The Fundamental Theorem of Arithmetic.) Every natural number greater than 1 has a unique factorization into one or more prime factors.

Theorem. (Completeness of $\mathbb{R}$.) Every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound.

Theorem. (Archimedean property of $\mathbb{R}$.) For any positive $a, b \in \mathbb{R}$, there is an $n \in \mathbb{N}$ such that $an > b$. (Equivalently: For any positive $x \in R$, there is an $n \in \mathbb{N}$ such that $n > x$.)

Theorem. (Heine-Borel Theorem.) Every open cover of a bounded closed interval in $\mathbb{R}$ has a finite subcover

Theorem. (Bolzano-Weirstrass Theorem.) Every bounded infinite subset of $\mathbb{R}$ has an accumulation point. (Stronger version: Every infinite subset of a bounded closed interval $[a, b]$ has an accumulation point in $[a, b]$.)

Theorem. (Properties of open sets.) Let (M, d) be a metric space. Then (1) $\varnothing$ and M are open subsets of M; (2) the union of any collection of open subsets of M is open; and (3) the intersection of any finite collection of open subsets of M is open.

Theorem. (Properties of compact sets.) Let K be a compact subset of some metric space. Then K is closed, K is bounded, and every infinite subset of K has an accumulation point in K.

Theorem. (Properties of limits.) [*see textbook for limits of sums, products, and quotients for limits of sequences and of functions.*]