Math 331, Fall 2019, Test 2 Information

The second in-class test will take place on Friday, November 8. It will cover Sections 2.3 through 3.6 in the textbook. There will be no questions specifically directed towards material from Sections 1.0 through 2.2, but you still need to know that material and might need it for the test (especially least upper bounds, greatest lower bounds, completeness, and limits). Questions on the test can include definitions, statements of theorems, short and long essay questions about concepts, concrete problems, and short proofs. Any proofs on the test will be ones that I believe should be straightforward, not requiring a great deal of thought.

Some things that you should know about for the test:

one-sided limits: $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$

continuity at a point; continuity on an open interval; continuity at an endpoint

uniform continuity on an interval

minimum/maximum/extreme value of a function on an set

minimum/maximum/extreme point of a function on a set

bounded function on an interval

differentiability of a function at a point; differentiability on an interval

partition of a closed bounded interval

upper and lower Riemann sums ($U(P;f)$ and $L(P;f)$)

infimum and supremum (inf and sup)

refinement of a partition

Riemann integrable function

the Riemann integral $\int_a^b f$

the Dirichlet function $D(x)$ — discontinuous at every point; not Riemann integrable

average value (or mean value) of an integrable function on an interval

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Some important definitions and theorems:

Definition. A function f is **continuous** at a point a if it is defined on an interval containing a and for every $\epsilon > 0$ there is a $\delta > 0$ such that $|x-a| < \delta$ implies $|f(x) - f(a)| < \epsilon$. [We can also define continuity from the left and from the right at a. For example, f is continuous from the right at a if it is defined on an interval of the form $[a, c)$ and for every $\epsilon > 0$, there is a $\delta > 0$ such that $0 \le x - a < \delta$ implies $|f(x) - f(a)| < \epsilon$.]

Definition. A function f is **uniformly continuous** on an interval I if for every $\epsilon > 0$, there is a $\delta > 0$ such that for every $x, y \in I$, $|x-y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

Definition. Let f be a function defined on an open interval containing a. We say that f is **differentiable** at a if the limit $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists. In that case, the value of the limit is denoted $f'(a)$.

Definition. A **partition** of $[a, b]$ is a sequence of points $P = \{x_o, x_1, \ldots, x_n\}$ such that $a = x_o < x_1 < \cdots < x_n = b$. If P and Q are partitions, we say Q is a **refinement** of P if Q contains every point that is in P.

Definition. Let f be a bounded function on $[a, b]$, and let $P = \{x_o, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. For $i = 1, 2, \ldots, n$, let $M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}$ and $m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$. We define the **upper Riemann sum** of f relative to the partition as $U(P; f) = \sum_{i=1}^{n} M_i(x_{i-1} - x_i)$, and we define the **lower Riemann sum** of f relative to the partition as $L(P; f) = \sum_{i=1}^{n} m_i(x_{i-1} - x_i)$.

Definition. We say that a function f is **integrable** on $[a, b]$ if it is bounded on $[a, b]$ and $\sup\{L(P; f) \mid P \text{ is a partition of } [a, b]\}$ is equal to $\inf\{U(P; f) \mid P \text{ is a partition of } [a, b]\}$. In that case, their common value is denoted $\int_a^b f$ and is called the **integral** of f on $[a, b]$.

Theorem. Let $a \in \mathbb{R}$ and f a function defined on an open interval containing a. Then $\lim_{x\to a} f(x)$ exists if and only if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist and are equal, and in that case, $\lim_{x\to a} f(x) = \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$.

Theorem. (Properties of limits.) Suppose $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist and $c \in R$. Then the following limits exist, with the given value:

$$\lim_{x\to a}(c \cdot f(x)) = c \cdot (\lim_{x\to a} f(x))$$
$$\lim_{x\to a}(f(x) + g(x)) = (\lim_{x\to a} f(x)) + (\lim_{x\to a} g(x))$$
$$\lim_{x\to a}(f(x) - g(x)) = (\lim_{x\to a} f(x)) - (\lim_{x\to a} g(x))$$
$$\lim_{x\to a}(f(x)g(x)) = (\lim_{x\to a} f(x)) \cdot (\lim_{x\to a} g(x))$$
$$\lim_{x\to a}(f(x)/g(x)) = (\lim_{x\to a} f(x))/(\lim_{x\to a} g(x)), \text{ if } \lim_{x\to a} g(x) \neq 0$$

Theorem. (Properties of continuity.) Suppose the functions f and g are both continuous at a. Then the following functions are also continuous at a: cf for any $c \in \mathbb{R}$, $f + g$, $f - g$, fg and, if $g(a) \neq 0$, f/g.

Theorem. (Continuity of compositions.) Suppose that the function g is continuous at a and that the function f is continuous at $g(a)$. Then the composition function $f \circ g$ is continuous at a.

Theorem. (IVT—Intermediate Value Theorem.) Suppose that the function f is continuous on $[a, b]$ and y is strictly between $f(a)$ and $f(b)$. Then there is a $c \in (a, b)$ such that $f(c) = y$.

Theorem. If the function f is a uniformly continuous function an interval I, then f is continous on I.

Theorem. (EVT—Extreme Value Theorem, *aka* Max-Min Theorem.) Suppose that f is a continuous function on the closed bounded interval $[a, b]$. Then f is bounded on $[a, b]$ and attains a minimum value and a maximum value on that interval. That is, there exist $c, d \in [a, b]$ such that $f(c) = glb\{f(x) \mid x \in [a, b]\}$ and $f(d) = lub\{f(x) \mid x \in [a, b]\}$. In other words, $f(c) \leq f(x) \leq f(d)$ for all $x \in [a, b]$.

Theorem. If f is defined on some open interval $[a, b]$ and has a maximum or minimum value for that interval at a point $c \in (a, b)$, and if $f'(c)$ exists, then $f'(c) = 0$,

Theorem. (Differentiability implies continuity.) If a function f is differentiable at a, then f is continuous at a.

Theorem. (Properties of the derivative.) [*Insert the constant multiple rule, sum rule, product rule, quotient rule, and chain rule here.*]

Theorem. (Rolle's Theorem.) Suppose that the function f is continuous on the interval $[a,b]$ and is differentiable on (a,b), and that $f(a) = f(b) = 0$. Then there is a $c \in [a,b]$ such that $f'(c) = 0$.

Theorem. (MVT—Mean Value Theorem.) Suppose that the function f is continuous on the interval $[a,b]$ and is differentiable on (a,b). Then there is a $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Theorem. (Generalized Mean Value Theorem.) Suppose that the functions f and g are continuous on $[a,b]$ and differentiable on (a,b). There there is a $c \in (a,b)$ such that $(g(b)-g(a))f'(x) = (f(b)-f(a))g'(c)$.

Theorem. Let f be a bounded function on $[a,b]$. Let P and Q be partitions of $[a,b]$ such that Q is a refinement of P. Then $L(P;f) \le L(Q;f) \le U(Q;f) \le U(P;f)$.

Theorem. Let f be a bounded function on $[a,b]$, and let P and Q be any two partitions of $[a,b]$. Then $L(Q;f) \le U(P;f)$. [This implies that the set $\{L(P;f) \,|\, P \text{ is a partition of } [a,b]\}$ is bounded above, that $\{U(P;f) \,|\, P \text{ is a partition of } [a,b]\}$ is bounded below, and that $\sup_P(\{L(P;f)\}) \le \inf_P(\{U(P;f)\})$.]

Theorem. Let f be a bounded function on $[a,b]$. Then f is Reimann integrable on $[a,b]$ if and only if for every $\epsilon > 0$, there is a partition P of $[a,b]$ such that $U(P;f) - L(P;f) < \epsilon$.

Theorem. If f is a continuous function on $[a,b]$, then f is Riemann integrable on $[a,b]$. Furthermore if P_n is the partition $\left\{a + i\frac{b-a}{n} \,|\, i = 0, 1, \ldots, n\right\}$, for $n \in \mathbb{N}$, then

$$\int_a^b f = \lim_{n\to\infty}\left(\sum_{i=1}^{n} f\left(a + i\tfrac{b-a}{n}\right)(x_i - x_{i-1})\right).$$

Theorem. If f is a non-decreasing function, or is a non-increasing function, on $[a,b]$, then f is Riemann integrable on $[a,b]$.

Theorem. (Linearity of the integral.) If f and g are Riemann integrable functions on $[a,b]$ and $c \in R$, then the functions cf and $f+g$ are Riemann integrable on $[a,b]$, and $\int_a^b cf = c\int_a^b f$, and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

Theorem. (Additivity of the integral.) If f is defined on $[a,b]$ and $a < c < b$, then f is Riemann integrable on $[a,b]$ if and only if f is Riemann integrable on $[a,c]$ and f is Riemann integrable on $[c,b]$, and in that case, $\int_a^b f = \int_a^c f + \int_c^b f$. [With the usual definitions of $\int_a^b f$ for $b = a$ and for $b < a$, this formula is valid even if c is not between a and b, as long as f is integrable on an interval that contains a, b, and c.]

Theorem. Let f be an integrable function on $[a,b]$, and define $F(x) = \int_a^x$ for $x \in [a,b]$. Then F is continuous on $[a,b]$. [In fact, F is uniformly continuous.]

Theorem. (First Fundamental Theorem of Calculus.) Suppose f is an integrable function on $[a,b]$ and g is a differentiable function on $[a,b]$ satisfying $g'(x) = f(x)$ for $x \in [a,b]$. Then $\int_a^b f = g(b) - g(a)$.

Theorem. (Mean Value Theorem for Integrals.) Suppose that f is a continuous function on $[a,b]$ Then there is a $c \in [a,b]$ such that $\int_a^b f = f(c)(b-a)$.

Theorem. (Second Fundamental Theorem of Calculus.) Let f be a continuous function on $[a,b]$, and define $F(x) = \int_a^x f$ for $x \in [a,b]$. Then F is differentiable on $[a,b]$ and $F'(x) = f(x)$ for $x \in [a,b]$. [In fact, if we only assume that f is continuous at some point $c \in [a,b]$, then F is differentiable at c, and $F'(c) = f(c)$.]