Your work should be submitted before the official final exam time,
1:30 PM, Thursday, December 10.

About the exam: Taking the final exam is optional. If you take it, you will have six grades, counting your homework grade twice. The lowest of those six grades will be dropped. Note that you cannot lower your grade for the course by taking the final exam; if your final exam grade is your lowest grade, it will be dropped.

This is a takehome exam, but with a short Zoom meeting to discuss your responses. As usual for this course, you can submit your work as an overleaf.com project or through Canvas. Your work is due by the officially scheduled final exam period, 1:30 PM on Thursday, December 10.

You can make an appointment to meet with me during the final exam period through the Canvas calendar, in the same way that you would for office hours. The Zoom meeting will use the same link as the one used for office hours, which can be found on the Canvas page for this course. If you cannot meet during the regular exam period for some reason, you should email me to set up an alternative time.

The work that you submit for this exam should be your own. You can use course materials, including the textbook, your notes, videos from online lectures, and posted solutions to homeworks. You should not use other books or material from the Internet. You can ask me questions about the exam, but you should not receive help on the exam from other students, your friends and family, or anyone else.

For the problems and proofs on this exam, you should present neatly written solutions. Write out your answers carefully, including explanations to justify your work when appropriate.

For the essay questions on this exam, you should write out clear and well-organized responses in complete sentences and paragraphs. The essay questions are meant to give you an opportunity to display your understanding of some central ideas from the course. Please type your answers to essay questions in LaTeX or in a word processing application. You can submit your essays as part of an overleaf.com project or through Canvas as separate documents. If you are submitting your work through Canvas, you must save your essays as a PDF file or as a Microsoft Word document.

If you are confused about any of the problems on the test, or if you have other questions, you can email me. I will also have some office hour times available in the Canvas calendar before the exam is due. While I won't give extensive help, I might be able to give some hints.

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1. [*From textbook problem 1.1.14, 6 points*] Prove that $\sqrt{2}+\sqrt{3}$ is irrational by filling in the details in this outline: Suppose $\sqrt{2}+\sqrt{3}$ is rational. Then $\sqrt{2}-\sqrt{3}$ is also rational. (Hint: Consider the product of these two numbers.) But then the sum of $\sqrt{2}+\sqrt{3}$ and $\sqrt{2}-\sqrt{3}$ is also rational, which leads to a contradiction.

2. [*6 points*] Let $f(x) = 3x^2 + 2x + 1$. Use the definition of derivative to show directly that for any $a \in \mathbb{R}$, $f'(a) = 6a + 2$.

3. [*6 points*] **(a)** Let $g(x) = \frac{1}{2}x|x|$. Show that $g'(x) = |x|$ for all x. (You can use the fact that $g'(0) = 0$, which was already shown on a homework problem. You can also use the facts that for $x > 0$, $g(x) = \frac{1}{2}x^2$, and for $x < 0$, $g(x) = -\frac{1}{2}x^2$. You do not have to use the definition of derivative here!)

(b) Use the result from part **(a)** and a single application of the First Fundamental Theorem of Calculus to find the value of $\displaystyle\int_{-5}^{10} |x|\, dx$.

4. [*6 points*] Let f and g be bounded functions on $[a, b]$. Suppose that $f(x) \leq g(x)$ for all $x \in [a, b]$. And let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition. Show that $U(P, f) \leq U(P, g)$.

5. [*6 points*] Find the interval of convergence of the following power series. (That is, find the radius of convergence, and determine whether the series converges or diverges at each endpoint. The ratio test or the root test can be used to find the radius of convergence.)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{5n}$$

6. [*8 points*] Suppose that f is a nondecreasing function on $[a, b]$. That is, for all $x_1, x_2 \in [a, b]$, if $x_1 < x_2$ then $f(x_1) \leq f(x_2)$. It is then true that the only kind of discontinuity that f can have on $[a, b]$ is jump discontinuities This problem asks you to provide an essential piece of the proof of that fact.

a) Let $c \in (a, b]$. Show that the set $\{f(x) \,|\, a \leq x < c\}$ is non-empty and bounded above and therefore has a least upper bound. [Hint: $f(c)$ is an upper bound for this set.]

b) Let $\lambda = lub\{f(x) \,|\, a \leq x < c\}$. Show that $\displaystyle\lim_{x \to c^-} f(x) = \lambda$. [Hint: This is similar to the proof of the Monotone Convergence Theorem. A picture will help!]

7. [*8 points*] Let f and g be functions that are uniformly continuous on $\mathbb{R}$. We already know that $f \cdot g$ is not necessarily uniformly continuous on $\mathbb{R}$: Let $f(x) = g(x) = x$. Then f and g are uniformly continuous on $\mathbb{R}$, but $(f \cdot g)(x) = f(x)g(x) = x^2$, which is not uniformly continuous on $\mathbb{R}$ by Example 2.6.1 in the textbook.

Now, suppose that f and g are uniformly continuous bounded functions on $\mathbb{R}$. So, there are constants $M_f > 0$ amd $M_g > 0$ such that $f(x) \leq M_f$ and $g(x) \leq M_g$ for all $x \in \mathbb{R}$. Prove that $f \cdot g$ is uniformly continuous on $\mathbb{R}$. [Hint: You will need to consider $|f(x)g(x) - f(y)g(y)|$. Try rewriting that as $|f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)|$.]

8. [*8 points*] One of the following is possible and one is impossible. Explain why the impossible one is impossible. For the one that is possible, give a specific series that satisfies the stated condition (and show that your example works!).

(a) A power series $\displaystyle\sum_{k=0}^{\infty} a_k x^k$ that converges at $x = 3$ but not at $x = 1$.

(b) A power series $\displaystyle\sum_{k=0}^{\infty} a_k x^k$ that converges at $x = 1$ but not at $x = 3$.

9. [*8 points*] Let $\sum_{k=1}^{\infty} a_k$ be a series. Define $\sum_{k=1}^{\infty} p_k$ and $\sum_{k=1}^{\infty} n_k$, the series of positive and of negative terms of this series, as follows:

$$p_k = \begin{cases} a_k & \text{if } a_k > 0 \\ 0 & \text{if } a_k \le 0 \end{cases} \qquad n_k = \begin{cases} 0 & \text{if } a_k > 0 \\ -a_k & \text{if } a_k \le 0 \end{cases}$$

Since $\sum_{k=1}^{\infty} p_k$ and $\sum_{k=1}^{\infty} n_k$ are non-negative series, they must either converge or diverge to $+\infty$. Prove that if $\sum_{k=1}^{\infty} a_k$ is conditionally convergent, then $\sum_{k=1}^{\infty} p_k$ and $\sum_{k=1}^{\infty} n_k$ both diverge to $+\infty$. [Hint: Clearly, $a_k = p_k - n_k$, so we also get identities like $n_k = p_k - a_k$ and $p_k = a_k + n_k$. Depending on how you approach this, you will probably need one at least one of these identities. Also, consider $|a_k|$.]

10. [*Textbook problem 4.5.7, 8 points*] **(a)** Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions defined on $[a, b]$. Assume that each f_n is bounded, that is there are constants M_n such that $|f_n(x)| \le M_n$ for all $x \in [a, b]$. Show: If $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f, then f must also be bounded.

(b) Show that the hypothesis of uniform convergence is necessary by finding a sequence of bounded functions that converges pointwise to a function that is not bounded. (Drawing a picture might help.)

11. [*10 points*] This is an essay question; see the instructions for essay questions on the first page of the exam. To answer this question properly, you will need something like a half-page essay single-spaced, or a full page double-spaced.

The relationship between derivatives and integrals is fundamental to Calculus. Write an essay discussing the relationship.

12. [*20 points*] This is an essay question; see the instructions for essay questions on the first page of the exam. To answer this question properly, you will need something close to a full page essay single-spaced, or two pages double-spaced.

A central concept in the study of the real numbers is *completeness*. One way of expressing the completeness of the real numbers is the least upper bound property. Completeness distinguishes the real numbers from similar number systems such as the rational numbers. Write an essay discussing the concept of completeness, what it means, how it was used in different parts of this course, and how it distinguishes the real numbers from the rational numbers.