This homework is due by the end of the day on Wednesday, Sept. 2
Many of the problems are from the exercises for Sections 1.1 and 1.2.

Problem 1. Prove by contradiction that $\log_{10} 2$ is irrational. (Hint: Suppose that $\log_{10} 2 = p/q$ for some integers p and q, where $q \neq 0$. Show that this would imply $2^q = 10^p$, and apply the Fundamental Theorem of Arithmetic.)

Problem 2. Prove that if a is irrational, then $\sqrt{a}$ is also irrational.

Problem 3. Determine whether each set is bounded above and if so find its least upper bound: $A = \{1 - \frac{1}{n} \,|\, n \in \mathbb{N}\}$; $B = \{\frac{n+1}{n} \,|\, n \in \mathbb{N}\}$; $C = [2, 9)$;
$D = \{1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{4}, 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}, 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}, \dots\}$;
$E = \{1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{3}, 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}, 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}, \dots\}$.

Problem 4. Let ε be any positive real number. Show that there is a natural number, n, such that $\frac{1}{n} < \varepsilon$. (Hint: Apply the Archimedian property of $\mathbb{R}$ to $\frac{1}{\varepsilon}$.)

Problem 5. Consider two sets of real numbers $A = \{a_1, a_2, a_3, \dots\}$ and $B = \{b_1, b_2, b_3, \dots\}$. Let C be the set $C = \{a_1 + b_1, a_2 + b_2, a_3 + b_3, \dots\}$.

(a) Suppose that μ_1 is an upper bound for A and μ_2 is an upper bound for B. Show that $\mu_1 + \mu_2$ is an upper bound for C.

(b) Now suppose that λ_1 is the least upper bound for A and λ_2 is the least upper bound for B. Give an example to show that $\lambda_1 + \lambda_2$ is not necessarily the least upper bound of C.

Problem 6. Let A and B be arbitrary non-empty, bounded-above sets of real numbers. Define $C = \{a + b \,|\, a \in A \text{ and } b \in B\}$. [Compare this to the previous problem, where C contains only the sums of corresponding elements of A and B; the C in this problem contains many elements that are not in the C in the preceding problem.]

(a) Suppose that λ_1 is the least upper bound for A and λ_2 is the least upper bound for B. Let $\lambda = \lambda_1 + \lambda_2$. Show that λ is an upper bound for C.

(b) Now, show that λ is the least upper bound for C. (Hint: Consider the last theorem in the third reading guide. Let $\varepsilon > 0$. Explain why there is a $a_o \in A$ such that $a_o > \lambda_1 - \frac{\varepsilon}{2}$ and a $b_o \in B$ such that $b_o > \lambda_2 - \frac{\varepsilon}{2}$. Use this to show $a_o + b_o > \lambda - \varepsilon$, and conclude that λ is the least upper bound for C.)

Problem 7. Let X be a non-empty, bounded-below subset of $\mathbb{R}$, and let μ be a lower bound for $\mathbb{R}$. Show that μ is the greatest lower bound of X if and only if for every $\varepsilon > 0$, there is an $x \in X$ such that $x < \mu + \varepsilon$. (Hint: Compare to the last theorem in the third reading guide.)

Problem 8. Prove that the intersection of two Dedekind cuts is again a Dedekind cut.