This homework is due by the end of the day on Friday, September 11
Problems are on Sections 1.3 and 1.4, not including Bolzano-Weirstrass.

Problem 1 (From Problem 1.3.7 in the textbook). Suppose that $(\mathbb{F}, +, \cdot)$ is a field, and $S \subseteq \mathbb{F}$. We say that S is a subfield of $\mathbb{F}$ if it is a field under the same addition and multiplication as $\mathbb{F}$. To show that S is a subfield of $\mathbb{F}$, it is enough to show that $0 \in S$, $1 \in S$, and S is closed under addition, multiplication, taking additive inverses, and taking multiplicative inverses. The other field axioms are automatically true for multiplication and addition in S, since they are already true in $\mathbb{F}$.

Let $\mathbb{Q}[\sqrt{2}] = \{r + s\sqrt{2} \mid r, s \in \mathbb{Q}\}$. Show that $\mathbb{Q}[\sqrt{2}]$ is a subfield of $\mathbb{R}$. (Hints: Note that r and s can be zero in $r+s\sqrt{2}$. To show that S is closed under taking multiplicative inverses of non-zero elements, you might need to show that if $r + s\sqrt{2} \neq 0$, then $r^2 - 2s^2 \neq 0$.)

Problem 2 (Problem 1.3.11 from the textbook). Let $(F, +, \cdot)$ be an ordered field. Use the definition of $x < y$ and the order axioms to prove the transitive property of $<$. That is, show that for any $a, b, c \in \mathbb{F}$, if $a < b$ and $b < c$, then $a < c$. [Note: Since $\mathbb{F}$ is not necessarily $\mathbb{R}$, you can't use common facts that you know about $\mathbb{R}$. You can only use the actual definition and axioms.]

Problem 3 (Problem 1.3.12 from the textbook. We proved that $1 > 0$ in class; this problem gives an alternative proof). Suppose $(\mathbb{F}, +, \cdot)$ is an ordered field, and $a \in \mathbb{F}$. Prove: If $a \neq 0$ then $a^2 > 0$. Conclude that $1 > 0$. [You can use Theorem 1.3.6, which says that $(-a)(-b) = ab$ in any field.]

Problem 4. **(a)** Let $\mathcal{O}_1, \mathcal{O}_2 \ldots, \mathcal{O}_k$ be some finite number of open subsets of $\mathbb{R}$. Prove that their intersection, $\bigcap_{i=1}^{k} \mathcal{O}_i$, is open. (Hint: Use the characterization of open that involves $\varepsilon > 0$. Start by taking arbitrary $x \in \bigcap_{i=1}^{k} \mathcal{O}_i$.) **(b)** Show that the intersection of an infinite number of open sets is not necessarily open by finding $\bigcap_{n=1}^{\infty} \big(-1 - \frac{1}{n}, 1 + \frac{1}{n} \big)$. (Justify your answer!)

Problem 5. Consider the **unbounded** closed interval $[0, \infty)$. Find an open cover of this interval that has no finite subcover. (Prove your answer!)

Problem 6 (Problem 1.4.3 from the textbook). Suppose that $\{\mathcal{O}_\alpha \mid \alpha \in A\}$ is an open cover of the interval $[0, 1)$. Suppose furthermore that $1 \in \bigcup_{\alpha \in A} \mathcal{O}_\alpha$. Prove that there is finite subcover of $[0, 1)$ from $\{\mathcal{O}_\alpha \mid \alpha \in A\}$. [This question tests your understanding of the proof of the Heine-Borel Theorem.]

Problem 7. Let $f(x)$ be a real-valued function that is defined on an interval I. We say that f is bounded above on I if there is a number M such that $f(x) < M$ for all $x \in I$.

Suppose that $f(x)$ is defined on the bounded, closed interval $[a, b]$. Suppose that for every $x \in [a, b]$, there is an $\varepsilon > 0$ such that f is bounded above on the inteval $(x - \varepsilon, x + \varepsilon)$. Use the Heine-Borel theorem to prove that f is bounded above on $[a, b]$. (Hint: Compare this an example about functions that was done in class.)