This homework is due by the end of the day on Friday, September 18

Problem 1 (Textbook problem 1.4.12a)**.** Suppose that λ is the least upper bound of some set S, and that λ is *not* in S. Prove that λ is an accumulation point of S. [Hint: For any $\varepsilon > 0$, there is a point $s \in S$ such that $\lambda - \varepsilon < s < \lambda$. Now use the definition of accumulation point to finish the proof.]

Problem 2 (Textbook problems 1.4.9 and 1.4.10)**.** **(a)** Prove lemma 1.4.5: If x is an accumulation point of a set S and if $\varepsilon > 0$, then there is an infinite number of points of S within distance ε of S. [Hint: Suppose that for some $\varepsilon > 0$, there were only a finite number of points, $s_1, s_2, \ldots, s_k$, of S within ε of x, but not equal to x. Let $\varepsilon' = \min(|s_1 - x|, |s_2 - x|, \ldots, |s_k - x|)$. Now, show that no $s \in S$ satisfies $0 < |s - x| < \varepsilon'$.] **(b)** Deduce that if S is a **finite** subset of $\mathbb{R}$, then S has no accumulation points. [This is trivially a corollary of the lemma.]

Problem 3. Prove directly from the epsilon-delta definition of limits, that $\lim\limits_{x \to 5} \frac{2x+4}{7} = 2$.

Problem 4. Show directly, without using the product rule for limits, that $\lim\limits_{x \to 3} x^3 = 27$. (Note that $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.)

Problem 5. Suppose that $\lim\limits_{x \to a} f(x) = L$ and $c \in \mathbb{R}$. Show, using the definition of limit, that $\lim\limits_{x \to a} cf(x) = cL$. [Be careful: $c = 0$ is a special case.]

Problem 6 (Textbook problem 2.2.9)**.** Suppose that $f(x) \leq 0$ for all x in some open interval containing a, except possibly at a. Suppose that $\lim\limits_{x \to a} f(x) = L$. Show that $L \leq 0$. [Hint: Assume instead that $L > 0$. Let $\varepsilon = L/2$ and derive a contradiction.] (Remark: A similar proof shows that if $f(x) \geq 0$ for all x near a, then $\lim\limits_{x \to a} f(x) \geq 0$, if the limit exists.)

Problem 7. Let's say that the sum, difference, and constant multiple rules for limits have already been proved, but that the product rule still needs to be proved.

(a) Suppose $\lim\limits_{x \to a} f(x) = L$. Show (without using the product rule) that $\lim\limits_{x \to a} f(x)^2 = L^2$.

(b) Use part (a) and the fact that $xy = \frac{1}{4}\big((x + y)^2 - (x - y)^2\big)$ to prove the product rule for limits.