This homework is due by the end of the day on Saturday, September 26

Problem 1. Suppose that $f(x)$ is defined and bounded on an open interval containing 0, except possibly at 0 itself. (That is, there is a number B such that $|f(x)| < B$ for all x in that interval, except possibly $x = 0$.) Show that $\lim_{x\to 0} xf(x) = 0$. [Hint: Use the Squeeze Theorem and the fact that $|x|$ is a continuous function.]

Problem 2. Suppose that $f(x)$ is a continuous function. Show that $|f(x)|$ is also continuous. [Hint: This is trivial. Apply known facts.]

Problem 3 (Textbook problem 2.5.7)**.** Suppose that f is continuous at a and that $f(a) > 0$. Show that there is a $\delta > 0$ such that $f(x) > 0$ for all x in the interval $(a - \delta, a + \delta)$. [Hint: Let $\varepsilon = f(a)$.]

Problem 4 (Textbook problem 2.6.7b)**.** Show that $p(x) = x^4 - x^3 + x^2 + x - 1$ has at least two roots in the interval $[-1, 1]$.

Problem 5. Prove the second half of the Extreme Value Theorem. That is, show that if f is continuous on a closed, bounded interval $[a, b]$, then there is a $x_o \in [a, b]$ such that $f(x) \geq f(x_o)$ for all $x \in [a, b]$. [Hint: An easy way to do this is to consider the function $g(x) = -f(x)$.]

Problem 6. Show that any linear function $f(x) = mx + b$ is uniformly continuous on $\mathbb{R}$.

Problem 7. Let $f(x) = \frac{1}{x}$.

(a) Show that for any $c > 0$, $f(x)$ is uniformly continuous on $[c, \infty)$,

(b) Show that $f(x)$ is not uniformly continuous on $(0, \infty)$.

Problem 8 (Textbook problem 2.6.12ab)**.** We say that a function f satisfies a **Lipschitz condition** if there is a positive real number M such that for all $x, y \in \mathbb{R}$, $|f(x) - f(y)| < M|x - y|$.

(a) Show that if f satisfies a Lipschitz condition, then f is uniformly continuous on $(-\infty, \infty)$.

(b) Show that $f(x) = (1 + |x|)^{-1}$ is uniformly continuous on $(-\infty, \infty)$ by showing that it satisfies a Lipschitz condition. [Hint: Recall that $\big||x| - |y|\big| \leq |x - y|$.]