Math 331 **Homework 5**

This homework on Metric Spaces, Section 1, is due by 11:59 PM on Saturday, October 3. Remember that there is a test on Monday, October 5.

Problem 1. Show that $[0, 1)$ is neither open nor closed in $\mathbb{R}$ (with its usual metric).

Problem 2. Let (M, d) be a metric space, and let $x \in M$. Show that the set $\{x\}$ is closed in (M, d) by showing that its complement, $M \setminus \{x\}$, is open. Conclude that any finite subset of a metric space is closed. [Hint for the second part: use Theorem 1.3.]

Problem 3. Find an infinite collection of closed subsets in the metric space $\mathbb{R}$ (with its usual metric), whose union is not closed. (We have already seen that $\{[-1 - \frac{1}{n}, 1 + \frac{1}{n}] \mid n \in \mathbb{N}\}$ is an infinite collection of open sets whose intersection is not open. That showed that the finiteness condition is required in Theorem 1.2, part 3. Your answer for this problem shows that the finiteness condition is required in Theorem 1.3, part 2.)

Problem 4. Let X be any non-empty set, and define $\delta\colon X \times X \to \mathbb{R}$ by $\delta(a, b) = \begin{cases} 0 & \text{if } a = b \\ 1 & \text{if } a \neq b \end{cases}$. Show that δ is a metric for X. It is called the **discrete metric** on X. Show that **every** subset of X is open in the metric space (X, δ). [Hint: What is $B^{\delta}_{1/2}(x)$?]

Problem 5. Let X be any non-empty, bounded subset of $\mathbb{R}$, and let λ be the least upper bound of X. Show that $\lambda \in \overline{X}$. That is, the least upper bound of X is an element of the closure of X. [Hint: By definition, $\overline{X}$ is the union of X with the set of all accumulation points of X, so consider the cases $\lambda \in X$ and $\lambda \notin X$.]

Problem 6. Consider the set $\mathscr{C}([a, b], \mathbb{R})$, the set of continuous, real-valued functions on the closed, bounded interval $[a, b]$. Define a metric σ on this set by

$$\sigma(f, g) = \int_a^b |f(x) - g(x)|\, dx$$

Show that σ is in fact a metric. (You will need some facts from Calculus about integrals, and you will need the triangle inequality for the standard metric on $\mathbb{R}$.) For a given $f \in \mathscr{C}([a, b], \mathbb{R})$ and $\varepsilon > 0$, try to describe what it means for a function g to be in $B^{\sigma}_{\varepsilon}(f)$. (Draw some pictures!)