This homework is due by 11:59 PM on Thursday, October 15.

Problem 1. Let $\{x_n\}_{n=1}^{\infty}$ be a convergent sequence in a metric space. Show that its limit is unique. That is, prove the following statement: if $\{x_n\}_{n=1}^{\infty}$ converges to y and $\{x_n\}_{n=1}^{\infty}$ converges to z, then $y = z$. [Hint: You can just imitate the proof for a real-valued function of a real variable.]

Problem 2. Let X be any set, and Consider the metric space (X, δ) where δ is the discrete metric, $\delta\colon X \times X \to \mathbb{R}$ by $\delta(a,b) = \begin{cases} 0 & \text{if } a = b \\ 1 & \text{if } a \neq b \end{cases}$. Suppose that $\{x_i\}_{i=1}^{\infty}$ is a **convergent** sequence in the metric space (X, δ). Show that there is a number N such that $x_N = x_{N+1} = x_{N+2} = \cdots$. [Hint: The number is the limit of the sequence.] (We say that the sequence is "eventually constant.")

Problem 3. Let (X, δ) be the discrete metric space from the previous exercise, and let (M, d) be any metric space. Show that **any** function $f\colon X \to M$ is continuous. (There are at least three possible proofs: using the definition of continuity, using Theorem 3.1, or using Theorem 3.3 from Metric Spaces Section 3.)

Problem 4. Define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} x & \text{if } x \leq 1 \\ x+1 & \text{if } x > 1 \end{cases}$. We know from calculus that f is not continuous at 1, so it must fail the continuity tests in Theorems 3.1 and 3.3 from Metric Spaces Section 3. Find a sequence $\{x_i\}_{i=1}^{\infty}$ that converges to 1, but $\{f(x_i)\}_{i=1}^{\infty}$ does not converge to $f(1)$. And find an open subset $\mathcal{O}$ of $\mathbb{R}$ such that $f^{-1}(\mathcal{O})$ is not open. (The values of the limits and of $f^{-1}(\mathcal{O})$ that you look at should be clear, and you do not need to prove that the values that you assert are correct.)

Problem 5 (Textbook problem 3.1.3a)**.** Even though $|x|$ is not differentiable at 0, show that the function $f(x) = x|x|$ is differentiable at 0, and find $g'(0)$.

Problem 6 (Textbook problem 3.2.8)**.** Use mathematical induction, the product rule, and the fact that $\frac{d}{dx}x = 1$ to prove that $\frac{d}{dx}x^n = nx^{n-1}$ for all $n \in \mathbb{N}$.

Problem 7. Recall that f satisfies a Lipschitz condition if there is a constant M such that $|f(b) - f(a)| \leq M|b - a|$ for all a, b, and any function that satisfies a Libschitz condition is uniformly continuous. Let f be a function that is differentiable everywhere, and suppose $f'(x) \leq M$ for all x, where M is some constant. Use the Mean Value Theorem to prove that $|f(b) - f(a)| \leq M|a - b|$ for all a, b. Conclude that f is uniformly continuous.

Problem 8 (Textbook problem 3.3.10)**.** A **fixed point** of a function is a point d such that $f(d) = d$. Suppose that f is differentiable everywhere and that $f'(x) < 1$ for all x. Show that there can be at most one fixed point for f. [Hint: Suppose that a and b are two fixed points of f. Apply the Mean Value Theorem to obtain a contradiction.]