This homework is due by 11:59 PM on Tuesday, November 3.

Problem 1. We showed that if f is integrable on $[a, b]$, then $|f|$ is also integrable on $[a, b]$. Now, suppose we know that $|g|$ is integrable on $[a, b]$. Is it necessarily true that g is integrable on $[a, b]$? [Hint: Consider a simple modification of the Dirichlet function.]

Problem 2. Suppose f is a continuous function on $[a, b]$ and $f(x) > 0$ for $x \in [a, b]$. Define $F(x) = \int_a^x f$. Prove that F is strictly increasing on $[a, b]$. [Hint: This is trivial, using two facts that we have proved.]

Problem 3. Suppose that f and g are continuously differentiable functions on $[a, b]$. So, f, g, f' and g' are all continuous. Prove the *Integration by Parts* formula

$$\int_a^b f(x)'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx$$

[Hint: One way to do this is to define, for $x \in [a, b]$, $P(x) = \int_a^x f(t)g'(t)dt$ and $Q(x) = f(t)g(t)\Big|_a^x - \int_a^x f'(t)g(t)dt = f(x)g(x) - f(a)g(a) - \int_a^x f'(t)g(t)dt$. Show that $P'(x) = Q'(x)$ and $P(a) = Q(a)$, and explain why this means $P(x) = Q(x)$ for all $x \in [a, b]$. Finally, use $P(b) = Q(b)$.]

Problem 4. Let f be the polynomial $f(x) = 3 + 2x^2 - 5x^3 + x^4$. Use Taylor's Theorem to write f as a polynomial in powers of $(x - 1)$. (That is, find the Taylor polynomial, $p_{4,1}(x)$, of degree 4 at 1 for f.)

Problem 5. Find the general Taylor polynomial at 0, $p_{n,0}(x)$, for the function $\ln(x + 1)$.

Problem 6 (from 4.2.14 from the textbook)**.** We have shown that the n^{th} Taylor polynomial for e^x at 0 is $p_{n,0}(x) = \sum_{k=1}^{n} \frac{1}{n!}x^n$. Show that e is irrational by using proof by contradiction. Suppose, for the sake of contradiction, that $e = \frac{p}{q}$ for some integers p and q.

(a) Use the Lagrange form of the remainder term from Taylor's Theorem to show that there is a $c \in [0, 1]$ such that $\frac{p}{q} - \left(\frac{1}{0!} + \frac{1}{1!} + \cdots + \frac{1}{n!}\right) = \frac{e^c}{(n+1)!}$.

(b) Multiply both sides of the equation in (a) by $n!$, and show that left side of the resulting equation is an integer when $n \geq q$.

(c) Show that the right side of the equation that you got in part (b) is not an integer when $n > e$. Conclude that e is irrational.